Ralf Schindler April 2018

Vopěnka and Balcar on generic extensions

I'd like to report on a result of Vopěnka and Balcar which I learned from Bukovský.

Theorem 1 (Vopěnka–Balcar) Let W be an inner model of V. Let $\mathbb{P}$ be a set in W, and let $g \subset \mathbb{P}$. The following are equivalent.

(1) There is $R \subset \mathbb{P} \times \mathbb{P}$, $R \in W$ such that

(a) for each $p \in g$, $\{q : pRq\} \subset \mathbb{P} \setminus g$, and

(b) for each $A \in W$, $A \subset \mathbb{P} \setminus g$, there is some $p \in g$ with $\{q : pRq\} \supset A$.

(In other words, $\{\{q : pRq\} : p \in g\}$ is a $\subset$-cofinal subset of $\mathcal{P}(\mathbb{P} \setminus g) \cap W$.)

(2) There is some $\leq \subset \mathbb{P}^2$ such that $(\mathbb{P}; \leq)$ is a partial order and g is $(\mathbb{P}; \leq)$-generic over W.

Proof: (2) $\Rightarrow$ (1). Let $R = \perp$, where $p \perp q$ iff p and q are incompatible w.r.t. $\leq$.

(a) is trivial.

(b) : Let $A \in W$, $A \subset \mathbb{P} \setminus g$. Let $A' = \{p : \exists q \in A \ \ p \leq q\}$. Then $A' \cap g = \emptyset$.

Let $p \in g$, $p \Vdash \check{A}' \cap \dot{g} = \emptyset$. Then $p \perp q$ for all $q \in A'$ (hence for all $q \in A$) : otherwise if $r \leq p, q$, let g' be $\mathbb{P}$-generic over W with $r \in g'$; then $A' \cap g' = \emptyset$, but $r \in A' \cap g'$. Contradiction.

Hence $A \subset \{q : q \perp p\}$.

Let us now prove $(1) \Rightarrow (2)$. We may as well assume that R as in (1) is symmetric and non-reflexive, as we may replace R by

$$(R \cup R^{-1}) \setminus \mathrm{id} \restriction \mathbb{P} .$$

Let us then define $p \leq q$ by $\{r : r R p\} \supset \{r : r R q\}$.

<u>Claim 1.</u> $p R q \Rightarrow p \perp q$ in the sense of $\leq$.

Proof of Claim 1: Suppose that $p R q$ and

$p \parallel q$ in the sense of $\leq$. Let $r \leq p, q$.

Then $p R q \Rightarrow p R r \Rightarrow r R p \Rightarrow r R r$ $\lightning$

(Claim 1)

Claim 2. $g \subset \mathbb{P}$ is a filter.

Proof of Claim 2: First let $\{p, q\} \subset g$. Then $\{r : pRr \vee qRr\} \in \mathcal{P}(\mathbb{P} \setminus g) \cap W$, so there is by (b) some $s \in g$ with $\{r : sRr\} \supset \{r : pRr\} \cup \{r : qRr\}$, i.e., $s \leq p, q$.

Now let $p \in g$, $p \leq q$. If $q \notin g$, then by (b) there is some $r \in g$ with rRq, hence $r \perp q$ by Claim 1. But then $r \perp p$ by $p \leq q$. However, $\{r, p\} \subset g$, so this contradicts the previous paragraph.

$\dashv$ (Claim 2)

Claim 3. g is $(\mathbb{P}; \leq)$-generic over W.

~~Let~~ Proof of Claim 3: If $A \subset \mathbb{P} \setminus g$ were a maximal antichain, $A \in W$, then by (b) there would be some $p \in g$ with $\{q : qRp\} \supset A$. By Claim 1, $q \perp p$ for all $q \in A$. $\lightning$

$\dashv$ (Claim 3)

We have shown Theorem 1. $\dashv$

Theorem 2 (Vopěnka) Let W be an inner model of V. Suppose that for each $A \in V$, $A \subset W$, there are X, Y such that $A = \bigcup X$, $X \subset Y$, $Y \in W$, $\overline{\overline{Y}} \leq \kappa$, where κ is an unctbl. cardinal. Then there is some poset $\mathbb{P} \in W$ of size $\leq \kappa$ and some g which is $\mathbb{P}$-generic over W s.t. $V = W[g]$.

Proof: Let $f : \theta \to OR$, $f \in V$. Let $f = \bigcup X$, $X \subset Y \in W$, $\overline{\overline{Y}} \leq \kappa$. Let $g : \theta \to V$ be defined by

$g(\xi) = \{\eta : \exists a \in Y\ (\eta \text{ is least s.t. } (\xi, \eta) \in a)\}$.

Then $f(\xi) \in g(\xi)$, $\overline{\overline{g(\xi)}} \leq \kappa$, for all ξ.

Hence by Bukovský's theorem there is some $\mathbb{Q} \in W$ and some h $\mathbb{Q}$-generic over W s.t. $V = W[h]$ and $\mathbb{Q}$ has the κ^+-c.c. Fix such $\mathbb{Q}, h$.

By our hypothesis, we may write

$h = \bigcup X$, $X \subset Y \in W$, $\overline{\overline{Y}} \leq \kappa$.

Let $\tau \in W^{\mathbb{Q}}$, $\tau^h = \mathcal{P}(Y \setminus X) \cap W$.

Let us define $R \subset Y \times Y$ as follows. For $a, b \in Y$ let $a \mathrel{R} b$ iff

$$\exists p \in a \cap \mathbb{Q} \ \exists x \in \mathcal{P}(Y) \cap W \left((p, \check{x}) \in \tau \wedge b \in Y \right)$$

It is easy to see that (a) + (b) of (1) in the statement of Theorem 1 are satisfied (with $\mathbb{P} = Y$, $g = X$).

But then there is by Theorem 1 some $\leq$ on Y s.t. X is $(Y; \leq)$-generic over W. Of course, ~~[illegible]~~ $h = \bigcup X \in W[X]$, so that $W[X] \supseteq W[h] = V$. As $X \in V$, $W[X] = V$.

$\dashv$