Research Statement

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Currently, my research focuses on topics in the general areas of K-theory of C*-algebras, coarse geometry, index theory and noncommutative geometry. More specifically, I investigate uniformly finite K-homology theory, which can be understood as an attempt to compute K-theory of uniform Roe C*-algebras, C*-algebras arising from coarse geometry. Another topic I focus on is studying controlled coarse homology theories and their connection to isoperimetric functions and amenability.

Coarse geometric ideas were introduced to mathematics by the celebrated Mostow's rigidity theorem, and subsequently popularized by Gromov (on the group theory side) and by John Roe (on the analytic side).

Given a metric space (X, d), topology focuses on small–scale properties of the space (for instance, metrics d and $\min(d, 1)$ generate the same topology, but the second metric makes the space bounded). Coarse geometry does "the opposite": metrics d and $\max(d, 1)$ yield the same "coarse structure". We disregard any local phenomena, and focus on the large–scale structure. This can be visualized as follows: imagine viewing the space from bigger and bigger distance. Any bounded set in the space becomes smaller and smaller, until finally "in the limit" it just becomes a point.

Roe C*-algebras and uniform Roe C*-algebras (also known as translation C*-algebras) are operator algebras associated to metric spaces, and reflect their large-scale (coarse) structure. Particularly interesting and important examples are Cayley graphs of finitely generated groups. The importance of Roe C*-algebras and uniform Roe C*-algebras comes from the fact that their K-theory serves as a receptacle for higher indices of elliptic operators. An algorithm to compute the K-theory of Roe C*-algebras is provided by the coarse Baum–Connes conjecture. The content of the conjecture is roughly that the analytic index map between the K-homology of a metric space and the K-theory of its Roe C*-algebra is an isomorphism. If true, the K-theory of Roe C*-algebras is computable by a cutting-and-pasting (Mayer–Vietoris) argument. The coarse Baum–Connes conjecture has various consequences, mainly in the field of topology and geometry. For instance, it implies the strong Novikov conjecture on the homotopy invariance of higher signatures; and non-existence of metrics of positive scalar curvature on certain manifolds. The field was pioneered by John Roe in his work on index theory on open manifolds. The coarse Baum–Connes conjecture was proved for a large class of spaces (those which embed coarsely into a Hilbert space) by G. Yu.

On the other hand, N. Higson found counterexamples to the conjecture related to certain expander graphs. However, many interesting questions are still open. For example, the coarse geometric Novikov conjecture (which states that the coarse Baum–Connes index map is injective) is open for many interesting spaces, and there are no known counterexamples.

My first topic of interest is defining and putting to use a uniform K-homology theory. The ultimate goal is to have a uniform version of the coarse Baum–Connes conjecture; that is, the isomorphism question about the index map from a uniform K-homology theory of a metric space into the K-theory of its uniform Roe C*-algebra. These groups tend to be rather large, but this feature makes it easier to detect higher indices of elliptic operators. Further interest comes from Elek's characterization of amenability in terms of K-theory of uniform Roe C*-algebras. Also, the uniformly finite homology theory of Block and Weinberger naturally fits into the picture: there should be a Chern character map from the uniformly finite K-homology theory into the uniformly finite homology.

I have been successful in defining a suitable candidate for a uniform K-homology theory, which is functorial under continuous and uniformly proper coarse maps and proving a Mayer-Vietoris sequence. Furthermore, I constructed an assembly map and proved an amenability criterion in accordance with Block–Weinberger's and Elek's criteria. Using the Mayer-Vietoris sequence, I also proved that if the space is a Cayley graph of a finitely generated torsion-free discrete group Γ, then its uniform K-homology is naturally isomorphic to $KK^{\Gamma}(\underline{E}\Gamma, \ell^{\infty}(\Gamma))$. This furthermore shows that the isomorphism question of the uniform assembly map for such such a group Γ is equivalent to the Baum–Connes conjecture for Γ with coefficients in $\ell^{\infty}\Gamma$. Hence the knowledge about the Baum–Connes conjecture with commutative (but not separable) coefficients provides a computation of uniform K-homology groups.

Let me now turn to another topic of interest: K-exactness. Study of C*-algebraic properties of the uniform Roe C*-algebras showed its usefulness in providing a link between a coarse geometric property of groups (the property A of Yu, which is a generalization of amenability) and an analytic property of the reduced group C*-algebras (exactness). K-exactness is a K-theoretic version of exactness. My research focuses on studying coarse geometric properties of metric spaces, which relate to K-exactness of their uniform Roe C*-algebras. Previous results related to this topic are those of Guentner, Kaminker and Ozawa; Higson; Roe; Skandalis; Tu and Yu; Brodzki, Niblo and Wright; Ulgen.

My result on K-exactness is an extension of a construction of Ozawa. It exhibits a certain class of expander graphs, whose uniform Roe C*-algebras are not K-exact.

Let me now turn to my last topic of interest: controlled coarse homology theories. Together with Piotr W. Nowak, we have defined this generalization of uniformly finite homology of Block and Weinberger. We characterized vanishing of a certain fundamental class in this homology in terms of an isoperimetric inequality. Furthermore, we proved that on any discrete group, at most linear control is needed for this class to vanish. We showed that Erschler's computation of isoperimetric invariants of some wreath products give interesting examples for our theory. As applications of controlled coarse homology, we characterized existence of primitives of the volume form with prescribed growth and showed that classes in our theory obstruct weighted Poincaré inequalities.

Let me summarize my future plans concerning these topics:

- Obtain a result on non-vanishing of higher index of certain elliptic operators using the index map from uniform K-homology.
- Investigate the uniform version of the coarse Baum–Connes conjecture. On one side, prove that the index map is an isomorphism under the assumption that the space has finite asymptotic dimension or property A. On the other side, give examples when the index map is not an isomorphism.
- Use surgery theory in connection with controlled coarse homology to prove results about distortion of diffeomorphisms and positive scalar curvature on open manifolds.

Beyond these specific topics, I am interested in coarse geometry in general, especially coarse geometry of finitely generated groups — geometric group theory; and in various topics in pure C*-algebra theory.