Change of the base field

Theorem 2.2.7. *Let E be an elliptic curve over a perfect field k, and let K/k be a field extension. Then $E_K := E \times_k \operatorname{Spec} K$ is an elliptic curve over K. Let $P \in E_K$ be a closed point whose image $Q \in E$ under the projection $\mathrm{pr}_E \colon E_K \to E$ is a closed point of E and let t be a uniformizing parameter of the discrete valuation ring $\mathcal{O}_{E,Q}$. Then $\mathrm{pr}^* t$ is a uniformizing parameter of the discrete valuation ring $\mathcal{O}_{E_K,P}$.*

Proof. We may assume that the curve $E = \mathrm{V}_{\mathbb{P}^2}(G)$ is defined by a Weierstraß equation $G \in k[X,Y,Z]$ with non-zero discriminant $\Delta(G)$. Then $E_K = V_{\mathbb{P}^2}(G \otimes 1)$ is defined by the Weierstraß equation $G \otimes 1 \in k[X,Y,Z] \otimes_k K = K[X,Y,Z]$ which is a prime element of $K[X,Y,Z]$ by Lemma 2.1.6. Therefore E_K is an integral K-scheme of dimension 1, in other words, a curve. It is proper over K by Blatt 6, Aufgabe 2. The discriminant of $G \otimes 1$ is $\Delta(G \otimes 1) = \Delta(G) \otimes 1$ and hence non-zero. Therefore E_K is normal and by Theorem 1.9.7 its genus is 1. Since $(0:1:0) \in E_K(K)$ we conclude that E_K is an elliptic curve over K.

To prove the second part we consider the exact sequence of $\mathcal{O}_{E,Q}$-modules

$$0 \longrightarrow \mathcal{O}_{E,Q} \xrightarrow{\ \cdot\, t\ } \mathcal{O}_{E,Q} \longrightarrow \kappa(Q) \longrightarrow 0\,. \tag{1}$$

Since E is of finite type over k and Q is a closed point, the residue field $\kappa(Q)$ is a finite separable field extension of k. By the Theorem of the Primitive Element we can write $\kappa(Q) = k[T]/(f)$ for a separable irreducible polynomial $f \in k[T]$. The polynomial $f \otimes 1 \in k[T] \otimes_k K = K[T]$ may decompose into a product $f \otimes 1 = f_1 \cdot \ldots \cdot f_r$ of irreducible polynomials $f_i \in K[T]$. But the f_i are pairwise relatively prime in $K[T]$, because if f_i and f_j have non-trivial greatest common divisor h, then all zeros α of h in an algebraic closure K^{alg} of K are zeros of both f_i and f_j and hence are multiple zeros of f in contradiction to the separability of f. Then the Chinese Remainder Theorem implies that $\kappa(Q) \otimes_k K \cong K[T]/(f \otimes 1) \cong \prod_{i=1}^r K[T]/(f_i)$ is a product of fields $K_i := K[T]/(f_i)$. From

$$\operatorname{Spec} \kappa(Q) \times_X X_K = \operatorname{Spec} \kappa(Q) \times_X X \times_k \operatorname{Spec} K = \operatorname{Spec} \kappa(Q) \times_k \operatorname{Spec} K = \operatorname{Spec}(\kappa(Q) \otimes_k K)$$

we see that the fiber $\operatorname{Spec} \kappa(Q) \times_X X_K$ of $\mathrm{pr}_X \colon X_K \to X$ above Q is isomorphic to the spectrum of $\prod_{i=1}^r K_i$, that is, to the disjoint union of the points $\operatorname{Spec} K_i$. One of these points is the point $P \in \operatorname{Spec} \kappa(Q) \times_X X_K \subset X_K$. Let $\mathfrak{m} \subset \kappa(Q) \otimes_k K$ be the maximal ideal corresponding to P. The closed immersion $\operatorname{Spec} \kappa(Q) \hookrightarrow \operatorname{Spec} \mathcal{O}_{E,Q}$ induces the closed immersion $\varphi \colon \operatorname{Spec}(\kappa(Q) \otimes_k K) \hookrightarrow \operatorname{Spec}(\mathcal{O}_{E,Q} \otimes_k K)$ and we let $\mathfrak{p} := \varphi^*(\mathfrak{m}) \subset \mathcal{O}_{E,Q} \otimes_k K$ be the prime ideal corresponding to the image of P.

We now tensor equation (1) over k with K. The resulting sequence

$$0 \longrightarrow \mathcal{O}_{E,Q} \otimes_k K \xrightarrow{\ \cdot\,(t \otimes 1)\ } \mathcal{O}_{E,Q} \otimes_k K \longrightarrow \kappa(Q) \otimes_k K \longrightarrow 0$$

remains exact, because the k-module K is a direct sum of copies of k indexed by a k-basis of K, and tensoring is compatible with taking direct sums. Localizing at the prime ideal $\mathfrak{p}$ of $\mathcal{O}_{E,Q} \otimes_k K$ and observing that $(\mathcal{O}_{E,Q} \otimes_k K)_{\mathfrak{p}} = \mathcal{O}_{E_K,P}$ and $(\kappa(Q) \otimes_k K)_{\mathfrak{p}} = \kappa(P)$ we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_{E_K,P} \xrightarrow{\ \cdot\,(t \otimes 1)\ } \mathcal{O}_{E_K,P} \longrightarrow \kappa(P) \longrightarrow 0\,.$$

This shows that $\mathrm{pr}^* t = t \otimes 1$ generates the maximal ideal of $\mathcal{O}_{E_K,P}$ as desired. $\square$

Some conclusions of the theorem are true in more generality. To explain this we consider k-schemes for a fixed field k which is not necessarily perfect. For a k-scheme X which is integral we denote by $k(X)$ the function field of X, that is the residue field $\kappa(\eta)$ at the generic point η of X.

Definition 2.2.8. A k-scheme X is called *geometrically integral* if $X \times_k \operatorname{Spec} K$ is integral for every field extension K/k. One makes the corresponding definition for the properties *irreducible, reduced, normal, regular* instead of integral.

Remark. By taking $K = k$ we see that every geometrically integral k-scheme is integral, and similarly for the other properties.

Theorem 2.2.9. *If k is perfect and X is locally of finite type over k then X is geometrically reduced if and only if X is reduced. The same is true for the properties normal and regular.*

Proof. [GW, Corollary 5.57] for "reduced" and [EGA, IV_2, Proposition 6.7.7] for "normal" and "regular". □

To discuss the property "geometrically integral" we make the following

Definition 2.2.10. Let K/k be a field extension. We say that

(a) *k is algebraically closed in K* if $\{f \in K \colon f \text{ is algebraic over } k\} = k$.

(b) *K is separable over k* if for every field extension L/k the tensor product $K \otimes_k L$ is a reduced ring.

Proposition 2.2.11 *If k is perfect (that is, if every finite field extension of k is separable) then every field extension K/k is separable in the sense of Definition 2.2.10(b).*

Proof. [Bos, §7.3, Korollar 7]. □

Proposition 2.2.12. *Let X be a k-scheme. Then the following assertions are equivalent.*

(a) X is geometrically integral.

(b) For every integral k-scheme Y the product $X \times_k Y$ is integral.

(c) X is integral, $k(X)$ is separable over k, and k is algebraically closed in $k(X)$.

(d) There exists an algebraically closed extension Ω of k such that $X \times_k \operatorname{Spec} \Omega$ is integral.

(e) For every finite extension L of k the product $X \times_k \operatorname{Spec} L$ is integral.

This proposition is formulated in [GW, Proposition 5.51]. Their proof includes several references to [Bou]. Therefore we give a different proof which is based on the following two lemmas.

Lemma 2.2.13. *Let k be a field and let X and Y be two integral k-schemes. Then $k(X) \otimes_k k(Y)$ is an integral domain if and only if $X \times_k Y$ is integral. If this is the case then $k(X \times_k Y) = \operatorname{Quot}(k(X) \otimes_k k(Y))$.*

Proof. Consider open subsets $\emptyset \neq \operatorname{Spec}(A) \subseteq X$ and $\emptyset \neq \operatorname{Spec}(B) \subseteq Y$. Then we have $k(X) = \operatorname{Quot}(A)$ and $k(Y) = \operatorname{Quot}(B)$. Moreover, the tensor product $A \otimes_k B$ is a subring of $\operatorname{Quot}(A) \otimes_k \operatorname{Quot}(B)$ and

$$\operatorname{Quot}(A) \otimes_k \operatorname{Quot}(B) \;=\; \big((A \smallsetminus \{0\}) \otimes 1\big)^{-1}\big(1 \otimes (B \smallsetminus \{0\})\big)^{-1}(A \otimes_k B)$$

is a localization of $A \otimes_k B$.

If $A \otimes_k B$ is an integral domain then also its localization $\operatorname{Quot}(A) \otimes_k \operatorname{Quot}(B)$ is an integral domain and $\operatorname{Quot}(A \otimes_k B) = \operatorname{Quot}(\operatorname{Quot}(A) \otimes_k \operatorname{Quot}(B))$.

Conversely if $\operatorname{Quot}(A) \otimes_k \operatorname{Quot}(B)$ is integral, then its subring $A \otimes_k B$ is also integral and again $\operatorname{Quot}(A \otimes_k B) = \operatorname{Quot}(\operatorname{Quot}(A) \otimes_k \operatorname{Quot}(B))$. It remains to show that $X \times_k Y$ is irreducible. We consider the morphism $\operatorname{Spec} \operatorname{Quot}(k(X) \otimes_k k(Y)) \to X \times_k Y$ which factors through any open subset $\operatorname{Spec}(A \otimes_k B) \subset X \times_k Y$ for A and B as above. The image of this morphism is a single point $\eta \in X \times_k Y$. That η is the generic point of $X \times_k Y$ can be tested locally on all the open sets $\operatorname{Spec}(A \otimes_k B)$. Since the homomorphism $A \otimes_k B \to \operatorname{Quot}(k(X) \otimes_k k(Y))$ is injective, η corresponds to the zero ideal in $A \otimes_k B$ and is indeed the generic point. $\square$

Lemma 2.2.14. *Let K/k be a field extension. Then the following assertions are equivalent.*

(a) *For every field extension L/k the tensor product $K \otimes_k L$ is an integral domain.*

(b) *For every finite field extension L/k the tensor product $K \otimes_k L$ is an integral domain.*

(c) *There is an algebraically closed extension Ω/k such that the tensor product $K \otimes_k \Omega$ is an integral domain.*

(d) *K is separable over k and k is algebraically closed in K.*

Proof. [Bos, §7.3, Satz 14] proves the equivalence of (a), (b) and (d). Clearly (a) implies (c). To see that (c) implies (b) we choose a k-imbedding $L \hookrightarrow \Omega$ and use that $K \otimes_k L$ is a subring of $K \otimes_k \Omega$. $\square$

Proof of Proposition 2.2.12. By Lemma 2.2.13 the statements in Proposition 2.2.12 can be phrased in terms of the integrality of tensor products $k(X) \otimes_k L$, where L is arbitrary in (a), $L = k(Y)$ in (b), $L = \Omega$ in (d), and L is finite over k in (e).

To see that (a) and (b) are equivalent we may take $Y = \operatorname{Spec} L$ in (b). Also (a), (c), (d) and (e) are equivalent by Lemma 2.2.14. $\square$

We end this section with an example that was used in the proof of Theorem 2.6.3(c). Let E and $\widetilde{E}$ be elliptic curves over a perfect field k. Then $k(E)$ and $k(\widetilde{E})$ are separable over k by Proposition 2.2.11. Moreover, k is algebraically closed both in $k(E)$ and in $k(\widetilde{E})$. Therefore E and $\widetilde{E}$ are geometrically integral over k. Let $K := k(E)$ and $\widetilde{K} := k(\widetilde{E})$. Then $\widetilde{E} \times_k E$ and $E_{\widetilde{K}} := \operatorname{Spec} \widetilde{K} \times_k E$ are integral schemes. Let $\emptyset \neq \operatorname{Spec} A \subset E$ be an open subset. Then $\operatorname{Spec}(\widetilde{K} \otimes_k A) \subset E_{\widetilde{K}}$ is open. Therefore, by Lemma 2.2.13 the function field of $\widetilde{E} \times_k E$ is

$$k(\widetilde{E} \times_k E) = \operatorname{Quot}(\widetilde{K} \otimes_k K) = \operatorname{Quot}(\widetilde{K} \otimes_k A)$$

which by definition equals the function field $\widetilde{K}(E_{\widetilde{K}})$ of $E_{\widetilde{K}}$.

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