Algebraic Geometry 2 | Prof. Dr. Urs Hartl
WiSe 2012/13 | Martin Brandenburg

Homework sheet 2

Due date: Monday, 22.10.12 at 12 noon

1. A topological space X is called *quasi-compact* if every open cover has a finite subcover. A scheme is called *quasi-compact* (*irreducible*, *connected*, etc.) if the underlying topological space enjoys this property. Prove the following:

   (a) Every affine scheme is quasi-compact.

   (b) A scheme is quasi-compact if and only if it has a finite open cover consisting of affine schemes.

   (c) $\star$ Give an example of a scheme which is not quasi-compact. In particular, it is not affine. *(4+2* points)*

2. Let U be an open subset of a scheme X. Prove that $U := (U, \mathcal{O}_X|_U)$ is again a scheme. *Hint.* Reduce to the special case $X = \mathrm{Spec}(A)$, $U = D_X(f)$ for $f \in A$. Prove $D_X(f) \cong \mathrm{Spec}(A_f)$. *(4 points)*

3. Let X be a scheme or more generally a locally ringed space. For $x \in X$ let $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the *residue field* at x. Let K be a field. Find a bijection between the sets $\mathrm{Hom}(\mathrm{Spec}(K), X) = \{\text{morphisms } \mathrm{Spec}(K) \to X\}$ and $\{(x, \alpha) \colon x \in X,\ \alpha \in \mathrm{Hom}(\kappa(x), K)\}$. *(3 points)*

4. Let X be a topological space, $Z \subseteq X$ closed and $x \in Z$. We call x a *generic point* of Z if $Z = \overline{\{x\}}$ holds. Prove the following:

   (a) If Z has a generic point, then Z is irreducible.

   (b) If X is a scheme, the converse is also true. In fact, the generic point is unique and contained in every open subset $U \subset X$ with $U \cap Z \neq \emptyset$.
   Hint. First do the case that X is affine.

   (c) The spectrum of an integral domain is irreducible; the generic point corresponds to the zero ideal. *(6 points)*