Algebraic Geometry 2 | Prof. Dr. Urs Hartl
WiSe 2012/13 | Martin Brandenburg

Homework sheet 3

Due date: Monday, 29.10.12 at 12 noon

1. Let k be an algebraically closed field. According to Satz 1.2.14 there is a k-scheme associated to the affine variety $\mathbb{A}^1(k)$ in the sense of AG1. Which one is it? What are the differences? Analyze the same for the projective line $\mathbb{P}^1(k)$. *(2 points)*

2. Describe all the closed subschemes of $\mathrm{Spec}(\mathbb{Z})$. Which ones are reduced, irreducible, integral resp. regular? *(4 points)*

3. Let X be a locally ringed space and A be a ring. Prove that
$$\mathrm{Hom}(X, \mathrm{Spec}(A)) \to \mathrm{Hom}(A, \mathcal{O}_X(X)),\ (f, f^{\#}) \mapsto f^{\#}_{\mathrm{Spec}(A)}$$
is bijective. *Hint.* Given a homomorphism $\varphi : A \to \mathcal{O}_X(X)$, define the map $f : X \to \mathrm{Spec}(A)$ by $f(x) = \{a \in A : \varphi(a)_x \in \mathfrak{m}_x\}$. *(4 points)*

4. Let X be an integral scheme. We denote by η the generic point of X and call $K := \mathcal{O}_{X,\eta}$ the *function field* of X. Prove the following:
   (a) An affine scheme $\mathrm{Spec}(A)$ is integral (resp. reduced) if and only if A is an integral domain (resp. reduced).
   (b) K is indeed a field, namely the field of fractions of A for every affine open subset $\emptyset \neq \mathrm{Spec}(A) \subseteq X$.
   (c) For every open $\emptyset \neq U \subseteq X$ the homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = K$ is *injective.*
   (d) $\star$ For every open $\emptyset \neq U \subseteq X$ we have $\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq K$.
   (6 + 2 points)*