Algebraic Geometry 2 | Prof. Dr. Urs Hartl
WiSe 2012/13 | Martin Brandenburg

Homework sheet 4

Due date: Monday, 5.11.12 at 12 noon

1. Let R be a PID. Write down all the prime ideals of $R[X]$ explicitly. For that purpose compute all the fibers of the natural morphism $\mathbb{A}^1_R \to \mathrm{Spec}(R)$.

   (4 points)

2. Consider the morphism $f : \mathrm{Spec}(\mathbb{Z}[i]) \to \mathrm{Spec}(\mathbb{Z})$.

   (a) Prove that f is finite.

   (b) Compute the fibers of f. Count their elements and describe these as prime ideals explicitly. Which fibers are reduced?

   You may use without proof that $X^2 + 1 \in \mathbb{F}_p[X]$ for a prime p is reducible if and only if $p = 2$ or $p \equiv 3 \pmod 4$.

   (4 points)

3. Let $\mathcal{P}$ be one of the properties finite / of finite type / locally of finite type. Prove that $\mathcal{P}$ is stable under composition and base change, i.e.:

   (a) When two morphisms $f : X \to Y$ and $g : Y \to Z$ satisfy $\mathcal{P}$, then the same is true for $g \circ f : X \to Z$.

   (b) Let $\pi : S' \to S$ be a morphism. If a morphism $f : X \to S$ satisfies $\mathcal{P}$, then so does the projection $p_2 : X \times_S S' \to S'$.

   Hint. First deal with the case that all schemes are affine.

   (4 points)

4. The n-dimensional affine space over an arbitrary scheme S is defined to be the S-scheme $\mathbb{A}^n_S := \mathbb{A}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} S$.

   (a) Prove that $\mathbb{A}^n_S \times_S \mathbb{A}^m_S \cong \mathbb{A}^{n+m}_S$.

   (b) Is the projection $p : \mathbb{A}^n_S \to S$ of finite type, or even finite?

   (c) $\star$ Prove: When S is an integral scheme, then so is $\mathbb{A}^n_S$.

   (4+2* points)

5. $\star$ Let k be an algebraically closed field and $n \geq 2$. The morphism of varieties $\mathbb{P}^1(k) \to \mathbb{P}^1(k), (x : y) \mapsto (x^n : y^n)$ induces a finite morphism of schemes $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$. Describe the fibers of f.

   (4* points)