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Homework sheet 7

Due date: Monday, 26.11.12 at 12 noon

1. Let X, Y be integral schemes with generic points η_X resp. η_Y. A morphism $f : X \to Y$ is called *dominant* when the image $f(X) \subseteq Y$ is dense. Prove that the following assertions are equivalent:

   (a) The morphism f is dominant.

   (b) For open subsets $V \subseteq Y$, $\emptyset \neq U \subseteq f^{-1}(V)$ the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective.

   (c) We have $f(\eta_X) = \eta_Y$.

   (d) We have $\eta_Y \in f(X)$. *(4 points)*

   In the following exercises k will be a field.

2. Let $A = k[x, y]/(y^2 - x^3)$ and $X = \mathrm{Spec}(A)$ be Neil's parabola. You already know from AG1 that $A = k[t^2, t^3] \subseteq k[t]$, where $t = \frac{y}{x}$. Determine

   (a) the normalization $\pi : \tilde{X} \to X$;

   (b) the fiber of π over the singular point $(x, y) \in X$.

   (c) $\star$ Illustrate (a) and (b) by a picture.

   (4+1* points)

3. Given $d \in \mathbb{N}$, find a morphism $\mathbb{P}^1_k \to \mathbb{P}^1_k$ of degree d.

   (4 points)

4. Let k be an algebraically closed field of characteristic $\neq 2, 3$ and consider $K := k(x)[y]/(y^2 - x^3 - x)$.

   (a) Prove that K/k is a finitely generated field extension with transcendence degree equal to 1.

   (b) Find the corresponding normal proper curve C/k with $k(C) = K$.
   Hint. Prove normality using the Jacobi matrix (AG1, Def. 4.1.1) on both affine charts.

   (c) Describe the morphism $C \to \mathbb{P}^1_k$ on closed points, which corresponds to the inclusion $k(x) \hookrightarrow K$.

   (4 points)