Algebraic Geometry 2
WiSe 2012/13

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Homework sheet 9

Due date: Monday, 10.12.12 at 12 noon

1. Let k be an algebraically closed field of characteristic $\neq 2, 3$. Consider the curve $C = V_{\mathbb{P}^2}(Y^2Z - X^3 - X^2Z - Z^3)$ from Example 1.7.13 with function field $k(C) = k(y)[x]/(x^3 + x^2 + 1 - y^2)$. Compute $\operatorname{div}(y)$.

   (4 points)

2. Let C be a normal proper curve over k.

   (a) Construct a homomorphism of abelian groups $\deg : \mathrm{Cl}(C) \to \mathbb{Z}$ with kernel $\mathrm{Cl}^0(C)$.

   (b) Prove that $\deg : \mathrm{Cl}(\mathbb{P}^1) \to \mathbb{Z}$ is an isomorphism.

   (c) In case that k is algebraically closed, prove that $\deg : \mathrm{Cl}(C) \to \mathbb{Z}$ is surjective.

   (4 points)

3. Prove the converse of Exercise 2: If k is algebraically closed and C is a normal proper curve over k such that $\deg : \mathrm{Cl}(C) \to \mathbb{Z}$ is an isomorphism, then $C \cong \mathbb{P}^1$.

   Hint. For two closed points $P, Q \in C$ there is some $\varphi \in k(C)$ such that $(P) - (Q) = \operatorname{div}(\varphi)$.

   (4 points)

4. Analoguous to Definition 1.8.8. one can define for an A-algebra B the B-module $\Omega_{B/A}$ with generators $d(b)$ for $b \in B$ and relations $d(a) = 0$ for $a \in A$ and $d(b + b') = d(b) + d(b')$, $d(b \cdot b') = b \cdot d(b') + b' \cdot d(b)$ for $b, b' \in B$.

   Prove the following: When $B = A[x_1, \ldots, x_n]$, then $\Omega_{B/A}$ is a free B-module of rank n. Write down a basis explicitly.

   Remark. The corresponding geometric statement is: The tangent bundle of the scheme $\mathbb{A}^n$ is trivial. As for the manifold $\mathbb{R}^n$.

   (4 points)