Algebraic Geometry 2
WiSe 2012/13

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Homework sheet 10

Due date: Monday, 17.12.12 at 12 noon

1. Let C be a normal proper curve over k and $D \in \mathrm{Div}(C)$. For $U = \emptyset$ define $\mathcal{O}_C(D)(U) = \{0\}$ and for $\emptyset \neq U \subseteq C$ open define

$$\mathcal{O}_C(D)(U) := \{f \in k(C)^* : \forall P \in U : \mathrm{ord}_P(f) + \mathrm{ord}_P(D) \geq 0\} \cup \{0\}.$$

   (a) Prove that $\mathcal{O}_C(D)$ is a sheaf of abelian groups on C with the property $\mathcal{O}_C(D)(C) = \mathcal{L}(D)$.

   (b) In fact $\mathcal{O}_C(D)$ is an $\mathcal{O}_C$-module, i.e. for all $U \subseteq C$ open $\mathcal{O}_C(D)(U)$ is an $\mathcal{O}_C(U)$-module and the restriction maps are linear.

(4 points)

2. Let k be an algebraically closed field and $d \in \mathbb{N}$ such that $d \in k^*$. Find a canonical divisor on the Fermat curve $V_{\mathbb{P}^2}(x^d + y^d - z^d)$.

(4 points)

3. Prove the Theorem of Riemann-Roch for $\mathbb{P}^1_k$ directly: For all $D \in \mathrm{Div}(\mathbb{P}^1_k)$ it holds $\ell(D) - \ell(K - D) = \deg(D) + 1 - g$, where $g = 0$ and $K = -2(1:0)$ is the canonical divisor.

   Hint. For $D = \sum_i m_i \cdot (P_i)$ with $P_i = (f_i) \in \mathrm{Spec}\, k[t] = \mathbb{P}^1_k \setminus \{(1:0)\}$ the multiplication with $\prod_i f_i^{m_i}$ is a k-linear isomorphism between $\mathcal{L}(D)$ und $\{f \in k[t] : \deg(f) \leq \deg(D)\}$

(4 points)

4. $\star$ Let C be a normal proper curve over $k = k^{\mathrm{alg}}$ of genus 0. Prove that $C \cong \mathbb{P}^1_k$. Do you know an analoguous statement about 1-dimensional manifolds?

   Hint. Use the Theorem of Riemann-Roch applied to the divisor (P), where $P \in C$ is a closed point.

($4^\star$ points)