Algebraic Geometry 2 Prof. Dr. Urs Hartl
WiSe 2012/13 Martin Brandenburg

Homework sheet 11

Due date: Monday, 7.1.13 at 12 noon

1. Assume $\mathrm{char}(k) \neq 2$. Let e_1, e_2, $e_3 \in k$ be pairwise distinct. The projective curve $V_{\mathbb{P}^2}(Y^2 Z - (X - e_1 Z)(X - e_2 Z)(X - e_3 Z))$ is normal (you do not have to prove this). Compute for $i = 1, 2, 3$ the principal divisors $\mathrm{div}\left(\frac{X}{Z} - e_i\right)$ and $\mathrm{div}\left(\frac{Y}{Z}\right)$, as well as the divisors $\mathrm{div}\left(d\frac{X}{Z}\right)$ and $\mathrm{div}\left(\frac{Z}{Y}\, d\frac{X}{Z}\right)$.

   (4 points)

2. Let $C \subseteq \mathbb{P}^2_k$ be a curve in Weierstraß normal form over a perfect field k of characteristic 2. Prove that the discriminant Δ vanishes if and only if C is singular. In that case there is only one singular point, which lies in $C(k)$. *Hint.* In the case $a_1 \neq 0$ one can assume $a_3 = a_4 = 0$ after some change of coordinates.

   (4 points)

3. Let C be a normal proper curve over k of genus $g = 2$. Prove that there is a surjective separable morphism $C \to \mathbb{P}^1_k$ of degree 2. Deduce that $k(C)/k(t)$ is a Galois extension of degree 2. *Hint.* First prove that there is some divisor $D \geq 0$ with $\deg(D) = \ell(D) = 2$.

   (4 points)

4. $\star$ The *genus* of a finitely generated field extension K/k of transcendence degree 1 with $\{f \in K\colon f \text{ is algebraic over } k\} = k$ is defined as the genus of the corresponding normal proper curve C/k with $k(C) = K$. Assume $\mathrm{char}(k) \neq 2$. What is the genus of $k(x)[y]/(y^2 + x^4 - 1)$?

   ($4^\star$ points)