

# On a Conjecture of Rapoport and Zink

Urs Hartl \*

September 15, 2010

## Abstract

In their book Rapoport and Zink constructed rigid analytic period spaces  $\mathcal{F}^{wa}$  for Fontaine's filtered isocrystals, and period morphisms from PEL moduli spaces of  $p$ -divisible groups to some of these period spaces. They conjectured the existence of an étale bijective morphism  $\mathcal{F}^a \rightarrow \mathcal{F}^{wa}$  of rigid analytic spaces and of a universal local system of  $\mathbb{Q}_p$ -vector spaces on  $\mathcal{F}^a$ . For Hodge-Tate weights  $n - 1$  and  $n$  we construct in this article an intrinsic Berkovich open subspace  $\mathcal{F}^0$  of  $\mathcal{F}^{wa}$  and the universal local system on  $\mathcal{F}^0$ . We conjecture that the rigid-analytic space associated with  $\mathcal{F}^0$  is the maximal possible  $\mathcal{F}^a$ , and that  $\mathcal{F}^0$  is connected. We give evidence for these conjectures and we show that for those period spaces possessing PEL period morphisms,  $\mathcal{F}^0$  equals the image of the period morphism. Then our local system is the rational Tate module of the universal  $p$ -divisible group and enjoys additional functoriality properties. We show that only in exceptional cases  $\mathcal{F}^0$  equals all of  $\mathcal{F}^{wa}$  and when the Shimura group is  $\mathrm{GL}_n$  we determine all these cases.

*Mathematics Subject Classification (2000):* 11S20, (14G35, 14L05, 14M15)

## Contents

<b>Introduction</b>	<b>2</b>
<b>1 The Conjecture of Rapoport and Zink</b>	<b>3</b>
<b>2 Fontaine's Rings</b>	<b>8</b>
<b>3 <math>\varphi</math>-Modules</b>	<b>10</b>
<b>4 Constructing <math>\varphi</math>-Modules from Filtered Isocrystals</b>	<b>13</b>
<b>5 The Minuscule Case</b>	<b>17</b>
<b>6 Relation with Period Morphisms</b>	<b>23</b>
<b>7 PEL period morphisms</b>	<b>29</b>
<b>8 When does weakly admissible imply admissible ?</b>	<b>36</b>
<b>9 Conjectures</b>	<b>42</b>

---

\*The author acknowledges support of the Deutsche Forschungsgemeinschaft in form of DFG-grant HA3006/2-1 and SFB 878.

<b>A Berkovich's Analytic Spaces</b>	<b>44</b>
<b>References</b>	<b>46</b>

## Introduction

Rapoport and Zink fix a reductive group  $G$  over  $\mathbb{Q}_p$ , an element  $b \in G(K_0)$  for  $K_0 := W(\mathbb{F}_p^{\text{alg}})[\frac{1}{p}]$ , and a conjugacy class  $\{\mu\}$  of cocharacters of  $G$ . They consider the projective variety  $\check{\mathcal{F}}$  over a local Shimura field  $\check{E}$  parametrizing the cocharacters belonging to this conjugacy class. They show that the set of weakly admissible cocharacters is a rigid analytic subspace  $(\check{\mathcal{F}}_b^{wa})^{\text{rig}}$  of  $\check{\mathcal{F}}$ , which is called a *period space*; see [RZ96, Proposition 1.36]. On [RZ96, p. 29] they conjecture the existence of an étale morphism  $(\check{\mathcal{F}}_b^a)^{\text{rig}} \rightarrow (\check{\mathcal{F}}_b^{wa})^{\text{rig}}$  of rigid analytic spaces which is bijective on rigid analytic points, and the existence of a tensor functor  $\mathcal{Y}$  from  $\mathbb{Q}_p$ -rational representations  $\rho : G \rightarrow \text{GL}(V)$  to local systems of  $\mathbb{Q}_p$ -vector spaces on  $(\check{\mathcal{F}}_b^a)^{\text{rig}}$ , such that at every point  $\mu$  of  $(\check{\mathcal{F}}_b^a)^{\text{rig}}$  the fiber of the local system  $\mathcal{Y}(\rho)$  is the crystalline Galois representation corresponding by the Colmez-Fontaine Theorem [CF00] to the filtration on the isocrystal  $(V \otimes_{\mathbb{Q}_p} K_0, \rho(b) \cdot \varphi)$  given by  $\rho \circ \mu$ . De Jong [dJ95] pointed out that to study local systems it is best to work in the category of Berkovich's  $\check{E}$ -analytic spaces rather than rigid analytic spaces. For convenience of the reader we review Berkovich's definition and the relation with rigid analytic spaces in Appendix A. We show that in fact  $(\check{\mathcal{F}}_b^{wa})^{\text{rig}}$  is the rigid analytic space associated with an open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^{wa}$  of  $\check{\mathcal{F}}$  (Proposition 9.1). In Section 1 we recall the definition of  $(\check{\mathcal{F}}_b^{wa})^{\text{rig}}$  and the precise formulation of the conjecture. We slightly strengthen the conjecture (in Conjecture 1.3) and explain its interpretation in terms of the étale fundamental group.

In this article we contribute to the conjecture in the situation where  $G$  has a representation  $\rho$  such that all Hodge-Tate weights of  $\rho \circ \mu$  are  $n - 1$  and  $n$  for some integer  $n$ . We construct an intrinsic open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^0$  of  $\check{\mathcal{F}}_b^{wa}$  and the universal local system on  $\check{\mathcal{F}}_b^0$ . We conjecture that  $\check{\mathcal{F}}_b^0$  is connected (Conjecture 5.5) and equals the largest possible  $\check{\mathcal{F}}_b^a$  as above (Conjecture 5.10). The construction proceeds as follows. With any analytic point  $\mu$  of  $\check{\mathcal{F}}$  (these are the ones of which  $\check{E}$ -analytic spaces consist; see Definition A.1) we associate in Section 4 a  $\varphi$ -module  $\mathbf{M}_\mu$  over the ring  $\widetilde{\mathbf{B}}_{\text{rig}}^\dagger$  and we let  $\check{\mathcal{F}}_b^0$  be the set of those  $\mu$  for which  $\mathbf{M}_\mu$  is isoclinic of slope zero. The ring  $\widetilde{\mathbf{B}}_{\text{rig}}^\dagger$ , which is a kind of maximal unramified extension of the Robba ring, is defined in Section 2 and the notion of  $\varphi$ -modules is recalled in Section 3. Our construction is inspired by Berger's [Ber09] construction which associates with any *rigid analytic point*  $\mu \in \check{\mathcal{F}}$  (the ones whose residue field  $\mathcal{H}(\mu)$  is *finite* over  $\check{E}$ ) a  $(\varphi, \Gamma)$ -module over the Robba ring. Due to the restriction to rigid analytic points Berger's approach works even without the above assumption on the Hodge-Tate weights. However, the techniques we use in Section 5, where we prove that  $\check{\mathcal{F}}_b^0$  is an open  $\check{E}$ -analytic subspace of  $\check{\mathcal{F}}$ , require in an essential way to work with  $\check{E}$ -analytic spaces. These also have points whose residue field is not finite over  $\check{E}$ . Moreover, they are topological spaces in the classical sense and we show that the complement  $\check{\mathcal{F}} \setminus \check{\mathcal{F}}_b^0$  is the image of a compact set under a continuous map (Theorem 5.6). It is worth noting that this crude map is in fact only continuous and not a morphism of  $\check{E}$ -analytic or rigid analytic spaces. The universal local system on  $\check{\mathcal{F}}_b^0$  is constructed in Theorem 5.8.

For Hodge-Tate weights  $-1$  and  $0$  Rapoport and Zink also study a period morphism  $\check{\pi} : \check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{F}}_b^{wa}$  starting at the  $\check{E}$ -analytic space  $\check{\mathcal{G}}^{\text{an}}$  associated with a PEL moduli space

$\check{\mathcal{G}}$  of  $p$ -divisible groups in a fixed isogeny class. The problem to determine the image of such period morphisms was already mentioned by Grothendieck [Gro71]. By the Colmez-Fontaine Theorem [CF00] it contains every classical rigid analytic point. When  $G = \mathrm{GL}(V)$  de Jong [dJ95, §§6 and 7] observed that the rational Tate module of the universal  $p$ -divisible group over  $\check{\mathcal{G}}^{\mathrm{an}}$  descends to a local system  $\mathcal{V}$  of  $\mathbb{Q}_p$ -vector spaces on the image  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}})$  of  $\check{\pi}$ ; compare Theorem 6.3 where we also show that  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}}) = \check{\mathcal{F}}_b^0$  and that  $\check{\mathcal{G}}^{\mathrm{an}}$  is identified with the space of  $\mathbb{Z}_p$ -lattices inside  $\mathcal{V}$ . This uses a recent result of Faltings [Fal10]. In the general PEL-situation we prove in Theorem 7.4 that  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}}) = \check{\mathcal{F}}_b^0$  and that the rational Tate-module of the universal  $p$ -divisible group over  $\check{\mathcal{G}}^{\mathrm{an}}$  induces a tensor functor from  $\mathbb{Q}_p$ -rational representations of  $G$  to local systems of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$  as in the conjecture of Rapoport and Zink. We conjecture that  $\check{\mathcal{F}}_b^0$  is the largest open  $\check{E}$ -analytic subset on which such a tensor functor exists (Conjectures 6.5 and 7.6) and we give some evidence for this in Section 9. On rigid analytic spaces the inclusion  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}}) \subset \check{\mathcal{F}}_b^{\mathrm{wa}}$  gives an étale morphism  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}})^{\mathrm{rig}} \rightarrow (\check{\mathcal{F}}_b^{\mathrm{wa}})^{\mathrm{rig}}$  which is bijective on classical rigid analytic points. It was noted in [RZ96] and [dJ95] as a peculiarity that this étale morphism is bijective on rigid analytic points but need not be an isomorphism. We give a natural explanation of this phenomenon. Indeed, we show that in general  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}}) = \check{\mathcal{F}}_b^0$  is strictly contained in  $\check{\mathcal{F}}_b^{\mathrm{wa}}$  (Example 5.7) and hence  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}})^{\mathrm{rig}} \rightarrow (\check{\mathcal{F}}_b^{\mathrm{wa}})^{\mathrm{rig}}$  in general is not an isomorphism (Proposition 1.4). For  $G = \mathrm{GL}_n$  we even compile a complete list of  $b \in \mathrm{GL}_n(K_0)$  for which  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}}) = \check{\mathcal{F}}_b^{\mathrm{wa}}$  in Theorem 8.3. The fact that the inclusion  $\check{\pi}(\check{\mathcal{G}}^{\mathrm{an}}) \subset \check{\mathcal{F}}_b^{\mathrm{wa}}$  may be strict while it induces a bijection on rigid analytic points by the Colmez-Fontaine Theorem [CF00] again shows that one must take into considerations also the points of  $\check{\mathcal{F}}_b^{\mathrm{wa}}$  whose residue field is not finite over  $\check{E}$ . In other words one should work with Berkovich's  $\check{E}$ -analytic spaces rather than rigid analytic spaces.

To end this introduction let us mention that some of the results presented here were announced in [Har08]. The ideas in this article are inspired by our analogous theory in equal characteristic [Har10], or [Har09, §6.2], where we were able to prove the analogue of the Rapoport-Zink Conjecture and the surjectivity of the period morphism onto  $\check{\mathcal{F}}_b^0$  for arbitrary Hodge-Tate weights. In the beginning we had hoped to be able to extend our approach here also to Hodge-Tate weights other than  $n-1$  and  $n$ , but we did not succeed. Not even the construction of the  $\varphi$ -module  $\mathbf{M}_\mu$  over  $\check{\mathbf{B}}_{\mathrm{rig}}^\dagger$ , let alone the openness result could be established along this line. The reason lies in the fact that for analytic points  $\mu \in \check{\mathcal{F}}$  whose residue field  $\mathcal{H}(\mu)$  is not finite over  $\check{E}$  the ring  $\check{\mathbf{B}}_{\mathrm{rig}}^\dagger$  is not a  $\mathcal{H}(\mu)$ -algebra (like Fontaine's field  $\mathbf{B}_{\mathrm{dR}}$  is not a  $\mathbb{C}_p$ -algebra). So Berger's construction could not be adapted.

*Acknowledgments.* The author would like to thank F. Andreatta, L. Berger, V. Berkovich, O. Brinon, J.-M. Fontaine, A. Genestier, U. Görtz, M. Kisin, V. Lafforgue, J. Pottharst, M. Rapoport, and E. Viehmann for many helpful discussions and for their interest in this work.

## 1 The Conjecture of Rapoport and Zink

In this section we recall the construction of Rapoport's and Zink's  $p$ -adic period spaces. We strengthen their conjecture mentioned above and interpret it in terms of the étale fundamental group. Let us first recall the definition of filtered isocrystals. We denote by  $\mathbb{F}_p^{\mathrm{alg}}$  an algebraic closure of the finite field with  $p$  elements and by  $K_0 := W(\mathbb{F}_p^{\mathrm{alg}})[\frac{1}{p}]$  the fraction field of the ring of Witt vectors over  $\mathbb{F}_p^{\mathrm{alg}}$ . Let  $\varphi = W(\mathrm{Frob}_p)$  be the Frobenius lift

on  $K_0$ .

**Definition 1.1.** An  $F$ -isocrystal over  $\mathbb{F}_p^{\text{alg}}$  is a finite dimensional  $K_0$ -vector space  $D$  equipped with a  $\varphi$ -linear automorphism  $\varphi_D$ . If  $L$  is a field extension of  $K_0$  and  $\text{Fil}^\bullet D_L$  is an exhaustive separated decreasing filtration of  $D_L := D \otimes_{K_0} L$  by  $L$ -subspaces we say that  $\underline{D} = (D, \varphi_D, \text{Fil}^\bullet D_L)$  is a *filtered isocrystal over  $L$* . The integers  $h$  for which  $\text{Fil}^{-h} D_L \neq \text{Fil}^{-h+1} D_L$  are called the *Hodge-Tate weights* of  $\underline{D}$ . We let  $t_N(\underline{D})$  be the  $p$ -adic valuation of  $\det \varphi_D$  (with respect to any basis of  $D$ ) and we let

$$t_H(\underline{D}) = \sum_{i \in \mathbb{Z}} i \cdot \dim_L \text{gr}_{\text{Fil}^\bullet}^i(D_L).$$

The filtered isocrystal  $\underline{D}$  is called *weakly admissible*<sup>1</sup> if

$$t_H(\underline{D}) = t_N(\underline{D}) \quad \text{and} \quad t_H(\underline{D}') \leq t_N(\underline{D}') \quad (1.1)$$

for any subobject  $\underline{D}' = (D', \varphi_D|_{D'}, \text{Fil}^\bullet D'_L)$  of  $\underline{D}$ , where  $D'$  is any  $\varphi_D$ -stable  $K_0$ -subspace of  $D$  equipped with the induced filtration  $\text{Fil}^i D'_L = D'_L \cap \text{Fil}^i D_L$  on  $D'_L := D' \otimes_{K_0} L$ .

To construct period spaces let  $G$  be a reductive linear algebraic group over  $\mathbb{Q}_p$ . Fix a conjugacy class  $\{\mu\}$  of cocharacters

$$\mu : \mathbb{G}_m \rightarrow G$$

defined over subfields of  $\mathbb{C}_p$  (the  $p$ -adic completion of an algebraic closure of  $K_0$ ). Let  $E$  be the field of definition of the conjugacy class. It is a finite extension of  $\mathbb{Q}_p$ . Two cocharacters in this conjugacy class are called *equivalent* if they induce the same weight filtration on the category  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  of finite dimensional  $\mathbb{Q}_p$ -rational representations of  $G$ . There is a projective variety  $\mathcal{F}$  over  $E$  whose  $\mathbb{C}_p$ -valued points are in bijection with the equivalence classes of cocharacters (from the fixed conjugacy class). Namely let  $V$  in  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  be any faithful representation of  $G$ . With a cocharacter  $\mu$  defined over a field  $L$  one associates the filtration  $\text{Fil}_\mu^i V_L := \bigoplus_{j \geq i} V_{L,j}$  of  $V_L := V \otimes_{\mathbb{Q}_p} L$  given by the weight spaces

$$V_{L,j} := \{ v \in V_L : \mu(z) \cdot v = z^j v \text{ for all } z \in \mathbb{G}_m(L) \}.$$

This defines a closed embedding of  $\mathcal{F}$  into a partial flag variety of  $V$

$$\mathcal{F} \hookrightarrow \text{Flag}(V) \otimes_{\mathbb{Q}_p} E, \quad (1.2)$$

where the points of  $\text{Flag}(V)$  with value in a  $\mathbb{Q}_p$ -algebra  $R$  are the filtrations  $F^i$  of  $V \otimes_{\mathbb{Q}_p} R$  by  $R$ -submodules which are direct summands such that  $\text{rk}_R \text{gr}_{F^\bullet}^i$  is the multiplicity of the weight  $i$  of the conjugacy class  $\{\mu\}$  on  $V$ .

Now let  $b \in G(K_0)$ . For any  $V$  in  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  one obtains an  $F$ -isocrystal  $(V \otimes_{\mathbb{Q}_p} K_0, b \cdot \varphi)$ . Its automorphism group contains the group from [RZ96, Proposition 1.12]

$$J(\mathbb{Q}_p) = \{ g \in G(K_0) : g^{-1} b \varphi(g) = b \}. \quad (1.3)$$

<sup>1</sup>This used to be the terminology until Colmez–Fontaine [CF00] showed that for  $L/K_0$  finite *weakly admissible implies admissible*. Since we consider also infinite extensions  $L/K_0$  for which the Colmez–Fontaine Theorem fails we stick to the old terminology.

A pair  $(b, \mu)$  with an element  $b \in G(K_0)$  and a cocharacter  $\mu : \mathbb{G}_m \rightarrow G$  defined over  $L \supset K_0$  is called *weakly admissible* if for some faithful representation  $\rho : G \hookrightarrow \mathrm{GL}(V)$  in  $\underline{\mathrm{Rep}}_{\mathbb{Q}_p} G$  the filtered isocrystal

$$\underline{D}_{b, \mu}(V) := (V \otimes_{\mathbb{Q}_p} K_0, \rho(b) \cdot \varphi, \mathrm{Fil}_{\mu}^{\bullet} V_L) \quad (1.4)$$

is weakly admissible. In fact this then holds for any  $V$  in  $\underline{\mathrm{Rep}}_{\mathbb{Q}_p} G$ ; see [RZ96, 1.18]. If  $(b, \mu)$  is weakly admissible and  $L/K_0$  is finite then the filtered isocrystal  $\underline{D}_{b, \mu}(V)$  is *admissible*, that is it arises from a crystalline Galois-representation  $\mathrm{Gal}(L^{\mathrm{alg}}/L) \rightarrow \mathrm{GL}(U)$  via Fontaine's covariant functor

$$\underline{D}_{b, \mu}(V) \cong \mathbf{D}_{\mathrm{cris}}(U) := (U \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})^{\mathrm{Gal}(L^{\mathrm{alg}}/L)}$$

which by the Colmez-Fontaine theorem [CF00] is an equivalence of categories from crystalline representations of  $\mathrm{Gal}(L^{\mathrm{alg}}/L)$  to weakly admissible filtered isocrystals over  $L$ . We write  $V_{\mathrm{cris}}(\underline{D}_{b, \mu}(V)) := U$  for Fontaine's covariant inverse functor. The assignment

$$\underline{\mathrm{Rep}}_{\mathbb{Q}_p} G \longrightarrow \underline{\mathrm{Rep}}_{\mathbb{Q}_p}(\mathrm{Gal}(L^{\mathrm{alg}}/L)), \quad V \mapsto V_{\mathrm{cris}}(\underline{D}_{b, \mu}(V)) \quad (1.5)$$

defines a tensor functor from  $\underline{\mathrm{Rep}}_{\mathbb{Q}_p} G$  to the category of continuous  $\mathrm{Gal}(L^{\mathrm{alg}}/L)$ -representations in finite dimensional  $\mathbb{Q}_p$ -vector spaces.

Let  $\check{E} = EK_0$  be the completion of the maximal unramified extension of  $E$ . In what follows we consider cocharacters  $\mu$  defined over complete extensions  $L$  of  $\check{E}$ . Let  $\check{\mathcal{F}}^{\mathrm{rig}}$  be the rigid analytic space over  $\check{E}$  associated with the variety  $\check{\mathcal{F}} = \mathcal{F} \otimes_E \check{E}$ . Rapoport and Zink define the *p-adic period space* associated with  $(G, b, \{\mu\})$  as

$$(\check{\mathcal{F}}_b^{wa})^{\mathrm{rig}} := \{ \mu \in \check{\mathcal{F}}^{\mathrm{rig}} : (b, \mu) \text{ is weakly admissible} \}.$$

It is acted on by  $J(\mathbb{Q}_p)$ . Rapoport and Zink show that  $(\check{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$  is an admissible open rigid analytic subspace of  $\check{\mathcal{F}}^{\mathrm{rig}}$ ; see [RZ96, Proposition 1.36]. For the proof they reduce to the case where  $b$  is *decent with some positive integer  $s$* , that is

$$b \cdot \varphi(b) \cdot \dots \cdot \varphi^{s-1}(b) = s\nu(p). \quad (1.6)$$

where  $\nu \in \mathrm{Hom}_{K_0}(\mathbb{G}_m, G) \otimes_{\mathbb{Z}} \mathbb{Q}$  is the slope quasi-cocharacter [RZ96, 1.7] and  $s$  satisfies  $s\nu \in \mathrm{Hom}_{K_0}(\mathbb{G}_m, G)$ . If  $b$  is decent with integer  $s$  then  $(\check{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$  has a natural structure of rigid analytic subspace of  $(\mathcal{F} \otimes_E E_s)^{\mathrm{rig}}$  over  $E_s = E \cdot W(\mathbb{F}_{p^s})[\frac{1}{p}]$  from which it arises by base change to  $\check{E}$ . In [RZ96, p. 29] Rapoport and Zink make a conjecture on the existence of a universal local system of  $\mathbb{Q}_p$ -vector spaces on  $(\check{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$  which we refine in 1.3 below.

Stimulated by [RZ96], the notion of local systems of  $\mathbb{Q}_p$ -vector spaces on rigid analytic spaces was studied by de Jong [dJ95, §4] who pointed out that this is best done working with Berkovich's  $\check{E}$ -analytic spaces rather than rigid analytic spaces. So let  $\check{\mathcal{F}}^{\mathrm{an}}$  be the  $\check{E}$ -analytic space associated with the variety  $\check{\mathcal{F}}$ . See Appendix A for the notion of  $\check{E}$ -analytic space and the relation with rigid analytic spaces. In fact the argument of [RZ96, Proposition 1.36] shows that there exists an open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^{wa}$  of  $\check{\mathcal{F}}^{\mathrm{an}}$  whose associated rigid analytic space is the period space  $(\check{\mathcal{F}}_b^{wa})^{\mathrm{rig}}$ ; we explain the proof of this fact in Proposition 9.1.

Local systems of  $\mathbb{Q}_p$ -vector spaces on an  $\check{E}$ -analytic space  $Y$  are related to the category  $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(\pi_1^{\acute{\mathrm{e}}\mathrm{t}}(Y, \bar{y}))$  of finite dimensional continuous  $\mathbb{Q}_p$ -linear representations of the étale fundamental group  $\pi_1^{\acute{\mathrm{e}}\mathrm{t}}(Y, \bar{y})$  of  $Y$  as follows. This group was defined by de Jong [dJ95].

**Proposition 1.2.** ([dJ95, Theorem 4.2].) *For any geometric base point  $\bar{y}$  of  $Y$  there is a natural  $\mathbb{Q}_p$ -linear tensor functor*

$$\omega_{\bar{y}} : \mathbb{Q}_p\text{-Loc}_Y \rightarrow \underline{\text{Rep}}_{\mathbb{Q}_p}(\pi_1^{\text{ét}}(Y, \bar{y}))$$

which assigns to a local system  $\mathcal{V}$  the  $\pi_1^{\text{ét}}(Y, \bar{y})$ -representation  $\mathcal{V}_{\bar{y}}$ . It is an equivalence if  $Y$  is connected.

In particular, for any point  $y \in Y(L)$  and geometric base point  $\bar{y}$  lying above  $y$ , any local system  $\mathcal{V}$  of  $\mathbb{Q}_p$ -vector spaces on  $Y$  gives rise to a Galois representation

$$\text{Gal}(L^{\text{alg}}/L) = \pi_1^{\text{ét}}(y, \bar{y}) \rightarrow \pi_1^{\text{ét}}(Y, \bar{y}) \rightarrow \text{GL}(\mathcal{V}_{\bar{y}}). \quad (1.7)$$

We denote this Galois representation by  $\mathcal{V}_y$ . Now we propose the following refinement of the conjecture of Rapoport and Zink [RZ96, p. 29]; see also [DOR, Conjecture 11.4.4].

**Conjecture 1.3.** *There exists a unique largest arcwise connected dense open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^a \subset \check{\mathcal{F}}_b^{wa}$  invariant under  $J(\mathbb{Q}_p)$  with  $\check{\mathcal{F}}_b^a(L) = \check{\mathcal{F}}_b^{wa}(L)$  for all finite extensions  $L/\check{E}$ , and a tensor functor  $\underline{\mathcal{V}}$  from  $\underline{\text{Rep}}_{\mathbb{Q}_p}G$  to the category  $\mathbb{Q}_p\text{-Loc}_{\check{\mathcal{F}}_b^a}$  of local systems of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^a$  with the following property:*

$$\begin{aligned} & \text{For any point } \mu \in \check{\mathcal{F}}_b^a(L) \text{ with } L/\check{E} \text{ finite, the tensor functor } \underline{\text{Rep}}_{\mathbb{Q}_p}G \rightarrow \\ & \underline{\text{Rep}}_{\mathbb{Q}_p}(\text{Gal}(L^{\text{alg}}/L)), V \mapsto V_{\text{cris}}(\underline{D}_{b,\mu}(V)) \text{ from (1.5) is isomorphic to} \\ & V \mapsto \underline{\mathcal{V}}(V)_{\mu} \text{ from (1.7) which associates with a representation } V \in \\ & \underline{\text{Rep}}_{\mathbb{Q}_p}G \text{ the fiber at } \mu \text{ of the corresponding local system } \underline{\mathcal{V}}(V). \end{aligned} \quad (1.8)$$

Note that any  $\check{E}$ -analytic space is locally arcwise connected by [Beh90, Theorem 3.2.1]. Moreover, any open  $\check{E}$ -analytic subspace of  $\check{\mathcal{F}}^{\text{an}}$  is paracompact by Lemma A.3.

Forgetting the action of  $\text{Gal}(L^{\text{alg}}/L)$ , Condition (1.8) implies that

$$\begin{aligned} & \text{for any point } \mu \in \check{\mathcal{F}}_b^a(L) \text{ with } L/\check{E} \text{ finite, the fiber functors } \underline{\text{Rep}}_{\mathbb{Q}_p}G \rightarrow \\ & (\mathbb{Q}_p\text{-vector spaces}), V \mapsto V_{\text{cris}}(\underline{D}_{b,\mu}(V)) \text{ and } V \mapsto \underline{\mathcal{V}}(V)_{\mu} \text{ are isomorphic.} \end{aligned} \quad (1.9)$$

But note that (1.9) for  $G = \text{GL}_n$  simply says that  $\dim_{\mathbb{Q}_p} \underline{\mathcal{V}}(V) = \dim_{\mathbb{Q}_p} V$  and also in general (1.9) is weaker than Condition (1.8). To derive the original conjecture of Rapoport and Zink from Conjecture 1.3 we in addition note the following

**Proposition 1.4.** *If Conjecture 1.3 holds, then the open immersion  $\check{\mathcal{F}}_b^a \subset \check{\mathcal{F}}_b^{wa}$  induces an étale morphism of rigid analytic spaces  $(\check{\mathcal{F}}_b^a)^{\text{rig}} \rightarrow (\check{\mathcal{F}}_b^{wa})^{\text{rig}}$  which is bijective on rigid analytic points. It is an isomorphism if and only if  $\check{\mathcal{F}}_b^a = \check{\mathcal{F}}_b^{wa}$ .*

*Proof.* The functor  $(\ )^{\text{rig}}$  takes étale morphisms to étale morphisms. The rest is a consequence of Theorem A.2 since  $\check{\mathcal{F}}_b^a$  and  $\check{\mathcal{F}}_b^{wa}$  are paracompact by Lemma A.3.  $\square$

The nature of such a morphism  $(\check{\mathcal{F}}_b^a)^{\text{rig}} \rightarrow (\check{\mathcal{F}}_b^{wa})^{\text{rig}}$  may seem somewhat obscure since it is a non-quasi-compact refinement of the Grothendieck topology on the same underlying set of points  $\check{\mathcal{F}}_b^a(\check{E}^{\text{alg}}) = \check{\mathcal{F}}_b^{wa}(\check{E}^{\text{alg}})$ . It is less mysterious on the level of  $\check{E}$ -analytic spaces because there one can see the points in the complement  $\check{\mathcal{F}}_b^{wa} \setminus \check{\mathcal{F}}_b^a$ ; compare Example 5.7 below.

After recalling some of Fontaine's rings and some facts on  $\varphi$ -modules in the next two sections we will construct in Sections 4 and 5 for Hodge-Tate weights  $n - 1$  and  $n$  an open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^0$  of  $\check{\mathcal{F}}_b^{wa}$  which is a candidate for the space searched for in Conjecture 1.3.

Let us end this section by explaining the significance of Conjecture 1.3 in terms of the étale fundamental group. Let  $\mu \in \check{\mathcal{F}}_b^a(L)$  be a point with values in a finite extension  $L/\check{E}$  and let  $\bar{\mu}$  be a geometric base point of  $\check{\mathcal{F}}_b^a$  lying above  $\mu$ . Then Rapoport and Zink [RZ96, 1.19] consider a cohomology class

$$cl(b, \mu) = \kappa(b) - \mu^\# \in H^1(\mathbb{Q}_p, G)$$

where  $\kappa(b) \in \pi_1(G)_{\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$  is the Kottwitz point [Kot97, §7] and  $\mu^\#$  is the image of  $\mu$  under the natural map  $X_*(T) \rightarrow \pi_1(G) \rightarrow \pi_1(G)_{\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$ . By the weak admissibility of the pair  $(b, \mu)$ , the difference  $\kappa(b) - \mu^\#$  lies in  $(\pi_1(G)_{\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)})_{\text{tors}}$  which Kottwitz [Kot97, 3.2] identifies with  $H^1(\mathbb{Q}_p, G)$ .

Let  $\omega_0$  be the forgetful fiber functor  $\underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow (\mathbb{Q}_p\text{-vector spaces})$ . The cohomology class  $cl(b, \mu)$  defines a  $G$ -torsor over  $\mathbb{Q}_p$  which by [DM82, Theorem 3.2] is of the form  $\text{Isom}^\otimes(\omega_0, \tilde{\omega})$  for a fiber functor  $\tilde{\omega} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow (\mathbb{Q}_p\text{-vector spaces})$ . Consider the image of  $cl(b, \mu)$  in  $H^1(\mathbb{Q}_p, G^{\text{ad}})$  and let  $\tilde{G}$  be the associated inner form of  $G$ . This yields a non-canonical isomorphism of  $\mathbb{Q}_p$ -group schemes  $\tilde{G} \cong \text{Aut}^\otimes(\tilde{\omega})$  by [DM82, Proposition 2.8] which we fix in the sequel.

**Corollary 1.5.** *Let  $Y \subset \check{\mathcal{F}}_b^{wa}$  be an open connected  $\check{E}$ -analytic subspace, and let  $\bar{\mu}$  be a geometric base point of  $Y$ . Then the set of isomorphism classes of tensor functors  $\underline{\mathcal{V}} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-Loc}_Y$  satisfying (1.9) is in bijection with the set of isomorphism classes of continuous group homomorphisms*

$$\pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p)$$

where two homomorphisms are isomorphic if they differ by composition with an automorphism of the  $\mathbb{Q}_p$ -group scheme  $\tilde{G}$ .

*Proof.* Let  $\underline{\mathcal{V}} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-Loc}_Y$  be a tensor functor and consider the fiber functor from  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  to  $(\mathbb{Q}_p\text{-vector spaces})$  obtained as the composition  $\omega_{\bar{\mu}} \circ \underline{\mathcal{V}}$  with the forgetful fiber functor  $\omega_{\bar{\mu}}$  on  $\underline{\text{Rep}}_{\mathbb{Q}_p}(\pi_1^{\text{ét}}(Y, \bar{\mu}))$  from Proposition 1.2. By [dJ95, Theorem 2.9] the fiber functors  $\omega_{\bar{\mu}}$  and  $\omega_{\bar{\mu}'}$  at any two geometric base points  $\bar{\mu}$  and  $\bar{\mu}'$  are isomorphic. So without changing the isomorphism class of  $\omega_{\bar{\mu}} \circ \underline{\mathcal{V}}$  we may assume that  $\bar{\mu}$  maps to a point  $\mu \in Y(L)$  with values in a finite extension  $L/\check{E}$ . Then by a theorem of Wintenberger [Win97, Corollary to Proposition 4.5.3] (see also [CF00, Proposition on page 4]) Condition (1.9) is equivalent to the assertion that the cohomology class of the  $G$ -torsor  $\text{Isom}^\otimes(\omega_0, \omega_{\bar{\mu}} \circ \underline{\mathcal{V}})$  equals  $cl(b, \mu) \in H^1(\mathbb{Q}_p, G)$ . This yields a non-canonical isomorphism of fiber functors  $\beta_0 : \tilde{\omega} \xrightarrow{\sim} \omega_{\bar{\mu}} \circ \underline{\mathcal{V}}$  by [DM82, Theorem 3.2], and a non-canonical isomorphism of  $\mathbb{Q}_p$ -group schemes  $\tilde{G} \cong \text{Aut}^\otimes(\tilde{\omega}) \cong \text{Aut}^\otimes(\omega_{\bar{\mu}} \circ \underline{\mathcal{V}})$ . Any element  $\gamma \in \pi_1^{\text{ét}}(Y, \bar{\mu})$  yields a tensor automorphism of  $\omega_{\bar{\mu}} \circ \underline{\mathcal{V}}$ . This defines a group homomorphism  $f : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p)$  which is unique up to composition with an automorphism of the group scheme  $\tilde{G}$ . Since for all  $\rho \in \underline{\text{Rep}}_{\mathbb{Q}_p} G$  the induced homomorphism  $\pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p) \rightarrow \text{GL}(\omega_{\bar{\mu}} \circ \underline{\mathcal{V}}(\rho))(\mathbb{Q}_p)$  is continuous, also  $f$  is continuous.

Conversely let  $f : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p)$  be a continuous group homomorphism. Then we define a tensor functor  $\underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \underline{\text{Rep}}_{\mathbb{Q}_p}(\pi_1^{\text{ét}}(Y, \bar{\mu}))$  by sending a representation  $\rho$  of  $G$  to the representation

$$\rho' : \pi_1^{\text{ét}}(Y, \bar{\mu}) \longrightarrow \text{GL}(\tilde{\omega}(\rho))(\mathbb{Q}_p), \quad \gamma \mapsto \tilde{\omega}(\rho)(f(\gamma)).$$

Here  $\tilde{\omega}(\rho)(f(\gamma))$  is the automorphism by which  $f(\gamma) \in \tilde{G}(\mathbb{Q}_p) \cong \text{Aut}^{\otimes}(\tilde{\omega})(\mathbb{Q}_p)$  acts on the  $\mathbb{Q}_p$ -vector space  $\tilde{\omega}(\rho)$ . Note that  $\rho'$  is continuous because  $\tilde{G}(\mathbb{Q}_p) \rightarrow \text{GL}(\tilde{\omega}(\rho))(\mathbb{Q}_p)$  is continuous. Let  $\underline{\mathcal{V}}(\rho) := \omega_{\bar{\mu}}^{-1}(\rho')$  be the local system of  $\mathbb{Q}_p$ -vector spaces on  $Y$  induced from  $\rho'$  via Proposition 1.2. This defines a tensor functor  $\underline{\mathcal{V}} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-Loc}_Y$ . The composition  $\omega_{\bar{\mu}} \circ \underline{\mathcal{V}}$  with the forgetful fiber functor  $\omega_{\bar{\mu}}$  equals  $\tilde{\omega}$ . Since the cohomology class of  $\text{Isom}^{\otimes}(\omega_0, \tilde{\omega})$  equals  $cl(b, \mu)$ , Wintenberger's result mentioned above shows that the tensor functor  $\underline{\mathcal{V}}$  satisfies condition (1.9). Clearly the assignments  $\underline{\mathcal{V}} \mapsto f$  and  $f \mapsto \underline{\mathcal{V}}$  are inverse to each other.  $\square$

**Remark 1.6.** In the situation of Corollary 1.5 any tensor functor  $\underline{\mathcal{V}} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-Loc}_Y$  induces a tower of étale covering spaces of  $Y$  with Hecke action. More precisely, let  $\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)$  be a compact open subgroup and let  $\check{\mathcal{E}}_{\tilde{K}}$  be the space representing  $\tilde{K}$ -level structures on  $\underline{\mathcal{V}}$ , that is residue classes modulo  $\tilde{K}$  of tensor isomorphisms  $(\beta : \tilde{\omega} \xrightarrow{\sim} \omega_{\bar{\mu}} \circ \underline{\mathcal{V}}) \bmod \tilde{K} \in \text{Isom}^{\otimes}(\tilde{\omega}, \omega_{\bar{\mu}} \circ \underline{\mathcal{V}})/\tilde{K}$  such that the class  $\beta\tilde{K}$  is invariant under the étale fundamental group of  $\check{\mathcal{E}}_{\tilde{K}}$ . Then  $\check{\mathcal{E}}_{\tilde{K}}$  is the étale covering space of  $Y$  corresponding to the discrete  $\pi_1^{\text{ét}}(Y, \bar{\mu})$ -set  $\text{Isom}^{\otimes}(\tilde{\omega}, \omega_{\bar{\mu}} \circ \underline{\mathcal{V}})/\tilde{K}$ . Any choice of a fixed tensor isomorphism  $\beta_0 \in \text{Isom}^{\otimes}(\tilde{\omega}, \omega_{\bar{\mu}} \circ \underline{\mathcal{V}})$  associates with  $\underline{\mathcal{V}}$  a representation  $\pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p)$  as in the proof of Corollary 1.5, and induces an identification of the  $\pi_1^{\text{ét}}(Y, \bar{\mu})$ -sets  $\text{Isom}^{\otimes}(\tilde{\omega}, \omega_{\bar{\mu}} \circ \underline{\mathcal{V}})/\tilde{K}$  and  $\tilde{G}(\mathbb{Q}_p)/\tilde{K}$ .

On the tower  $(\check{\mathcal{E}}_{\tilde{K}})_{\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)}$  the group  $\tilde{G}(\mathbb{Q}_p)$  acts via Hecke correspondences: Let  $g \in \tilde{G}(\mathbb{Q}_p)$  and let  $\tilde{K}, \tilde{K}' \subset \tilde{G}(\mathbb{Q}_p)$  be compact open subgroups. Then the Hecke correspondence  $\pi(g)_{\tilde{K}, \tilde{K}'}$  is given by the diagram

$$\begin{array}{ccc} & \check{\mathcal{E}}_{\tilde{K}' \cap g^{-1}\tilde{K}g} & \beta \bmod (\tilde{K}' \cap g^{-1}\tilde{K}g) \\ & \swarrow \quad \searrow & \swarrow \quad \searrow \\ \check{\mathcal{E}}_{\tilde{K}'} & \dashleftarrow \quad \dashrightarrow & \beta \bmod \tilde{K}' \\ & & \beta g^{-1} \bmod \tilde{K} \end{array}$$

The whole construction does not depend on the choice of the base point  $\bar{\mu}$  by [dJ95, Theorem 2.9] and hence also applies if  $Y$  is not connected.

## 2 Fontaine's Rings

We recall from Colmez [Col09] some of the rings used in  $p$ -adic Hodge theory. Let  $\mathcal{O}_L$  be a complete valuation ring of rank one which is an extension of  $\mathbb{Z}_p$  and let  $L$  be its fraction field. It is not assumed to be discrete. Let  $v_p$  be the valuation on  $\mathcal{O}_L$  which we assume to be normalized so that  $v_p(p) = 1$ . In  $p$ -adic Hodge theory it is usually assumed that  $\mathcal{O}_L$  is discretely valued with perfect residue field. However, in Section 4 we need the more general situation introduced here. Let  $C$  be the completion of an algebraic closure  $L^{\text{alg}}$  of  $L$  and let  $\mathcal{O}_C$  be the valuation ring of  $C$ . For  $n \in \mathbb{N}$  let  $\varepsilon^{(n)}$  be a primitive  $p^n$ -th root of unity chosen in such a way that  $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$  for all  $n$ . We define

$$\tilde{\mathbf{E}}^+ := \tilde{\mathbf{E}}^+(C) := \{x = (x^{(n)})_{n \in \mathbb{N}_0} : x^{(n)} \in \mathcal{O}_C, (x^{(n+1)})^p = x^{(n)} \text{ for all } n\}.$$

Fix the elements  $\varepsilon := (1, \varepsilon^{(1)}, \varepsilon^{(2)}, \dots)$  and  $\bar{\pi} := \varepsilon - 1$  of  $\tilde{\mathbf{E}}^+$ . With the multiplication  $xy := (x^{(n)}y^{(n)})_{n \in \mathbb{N}_0}$ , the addition  $x + y := (\lim_{m \rightarrow \infty} (x^{(m+n)} + y^{(m+n)})^{p^m})_{n \in \mathbb{N}_0}$ , and the valuation  $v_{\mathbf{E}}(x) := v_p(x^{(0)})$ ,  $\tilde{\mathbf{E}}^+(C)$  becomes a complete valuation ring of rank one with algebraically closed fraction field, called  $\tilde{\mathbf{E}} := \tilde{\mathbf{E}}(C)$ , of characteristic  $p$ . Next we define

$$\begin{aligned} \tilde{\mathbf{A}}^+ &:= \tilde{\mathbf{A}}^+(C) := W(\tilde{\mathbf{E}}^+(C)) \text{ and} \\ \tilde{\mathbf{A}} &:= \tilde{\mathbf{A}}(C) := W(\tilde{\mathbf{E}}(C)) \text{ the rings of Witt vectors,} \\ \tilde{\mathbf{B}}^+ &:= \tilde{\mathbf{B}}^+(C) := \tilde{\mathbf{A}}^+(C)[\frac{1}{p}] \text{ and} \\ \tilde{\mathbf{B}} &:= \tilde{\mathbf{B}}(C) := \tilde{\mathbf{A}}(C)[\frac{1}{p}] \text{ the fraction field of } \tilde{\mathbf{A}}(C). \end{aligned}$$

By Witt vector functoriality there is the Frobenius lift  $\varphi := W(\text{Frob}_p)$  on the later rings. For  $x \in \tilde{\mathbf{E}}(C)$  we let  $[x] \in \tilde{\mathbf{A}}(C)$  denote the Teichmüller lift. Set  $\pi := [\varepsilon] - 1$ . If  $x = \sum_{i=0}^{\infty} p^i [x_i] \in \tilde{\mathbf{A}}(C)$  then we set  $w_k(x) := \min\{v_{\mathbf{E}}(x_i) : i \leq k\}$ . For  $r > 0$  let

$$\begin{aligned} \tilde{\mathbf{A}}^{(0,r]} &:= \tilde{\mathbf{A}}^{(0,r]}(C) := \{x \in \tilde{\mathbf{A}}(C) : \lim_{k \rightarrow +\infty} w_k(x) + \frac{k}{r} = +\infty\}, \\ \tilde{\mathbf{B}}^{(0,r]} &:= \tilde{\mathbf{B}}^{(0,r]}(C) := \tilde{\mathbf{A}}^{(0,r]}(C)[\frac{1}{p}], \\ \tilde{\mathbf{B}}^\dagger &:= \tilde{\mathbf{B}}^\dagger(C) := \bigcup_{r \rightarrow 0} \tilde{\mathbf{B}}^{(0,r]}(C). \end{aligned}$$

One has  $\tilde{\mathbf{A}}^+(C) \subset \tilde{\mathbf{A}}^{(0,r]}(C) \subset \tilde{\mathbf{A}}^{(0,s]}(C)$  for all  $r \geq s > 0$ . On  $\tilde{\mathbf{B}}^{(0,r]}(C)$  there is a valuation defined for  $x = \sum_{i \gg -\infty} p^i [x_i]$  as

$$v^{(0,r]}(x) := \min\{w_k(x) + \frac{k}{r} : k \in \mathbb{Z}\} = \min\{v_{\mathbf{E}}(x_i) + \frac{i}{r} : i \in \mathbb{Z}\}.$$

Now one defines  $\tilde{\mathbf{B}}^{[0,r]}(C)$  as the Fréchet completion of  $\tilde{\mathbf{B}}^{(0,r]}(C)$  with respect to the family of semi-valuations  $v^{[s,r]}(x) := \min\{v^{(0,s]}(x), v^{(0,r]}(x)\}$  for  $0 < s \leq r$ . This means in concrete terms that a sequence of elements  $x_n \in \tilde{\mathbf{B}}^{(0,r]}(C)$  converges in  $\tilde{\mathbf{B}}^{[0,r]}(C)$  if and only if  $\lim_{n \rightarrow \infty} v^{[s,r]}(x_{n+1} - x_n) = +\infty$  for all  $0 < s \leq r$ . Also if  $r \geq s$  we let  $\tilde{\mathbf{B}}^{[s,r]}(C)$  be the completion of  $\tilde{\mathbf{B}}^{(0,r]}(C)$  with respect to  $v^{[s,r]}$ . We view  $\tilde{\mathbf{B}}^{[0,r]}(C)$  as a subring of  $\tilde{\mathbf{B}}^I(C)$  for any closed subinterval  $I \subset (0, r]$ . Let

$$\tilde{\mathbf{B}}_{\text{rig}}^\dagger := \tilde{\mathbf{B}}_{\text{rig}}^\dagger(C) := \bigcup_{r \rightarrow 0} \tilde{\mathbf{B}}^{[0,r]}(C).$$

The morphism  $\varphi$  extends to bicontinuous isomorphisms of rings  $\tilde{\mathbf{B}}^{[0,pr]}(C) \rightarrow \tilde{\mathbf{B}}^{[0,r]}(C)$  and  $\tilde{\mathbf{B}}^{[ps,pr]}(C) \rightarrow \tilde{\mathbf{B}}^{[s,r]}(C)$  defining an automorphism of  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger(C)$ . The rings  $\tilde{\mathbf{B}}^{[0,r]}(C)$  and  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger(C)$  were studied in detail by Berger [Ber02, §2] who called the first  $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger, (p-1)/pr}$  instead. Note that the ring  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger(C)$  is denoted  $\Gamma_{\text{an,con}}^{\text{alg}}$  by Kedlaya [Ked05, Definition 2.4.8, Remark 2.4.13] and is an integral domain.

There is a homomorphism  $\theta : \tilde{\mathbf{B}}^{(0,1]}(C) \rightarrow C$  sending  $\sum_{i \gg -\infty} p^i [x_i]$  to  $\sum_{i \gg -\infty} p^i x_i^{(0)}$  which extends by continuity to  $\tilde{\mathbf{B}}^{[0,1]}(C)$  and even to  $\tilde{\mathbf{B}}^{[1,1]}(C)$ . The series

$$t := \log[\varepsilon] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \pi^n \in \tilde{\mathbf{B}}^{[0,1]}(C)$$

satisfies  $\varphi(t) = pt$  and  $t\tilde{\mathbf{B}}^{[1,1]}(C) = \ker(\theta : \tilde{\mathbf{B}}^{[1,1]}(C) \rightarrow C)$ . Let

$$\mathbf{B}_{\text{cris}}^+ := \mathbf{B}_{\text{cris}}^+(C) \text{ be the } p\text{-adic completion of } \tilde{\mathbf{B}}^+(C) \left[ \frac{w^n}{n!} : w \in \ker \theta, n \in \mathbb{N} \right] \text{ and}$$

$$\mathbf{B}_{\text{cris}} := \mathbf{B}_{\text{cris}}(C) := \mathbf{B}_{\text{cris}}^+(C) \left[ \frac{1}{t} \right].$$

The morphism  $\varphi$  extends to endomorphisms of  $\mathbf{B}_{\text{cris}}^+(C)$  and  $\mathbf{B}_{\text{cris}}(C)$ . Let

$$\tilde{\mathbf{B}}_{\text{rig}}^+ := \tilde{\mathbf{B}}_{\text{rig}}^+(C) := \bigcap_{n \in \mathbb{N}_0} \varphi^n \mathbf{B}_{\text{cris}}^+(C).$$

One easily sees that  $\tilde{\mathbf{B}}_{\text{rig}}^+(C) \subset \tilde{\mathbf{B}}^{[0,r]}(C)$  for any  $r > 0$ . More precisely,  $\tilde{\mathbf{B}}^{[0,1]}(C)$  equals the  $p$ -adic completion of  $\tilde{\mathbf{B}}_{\text{rig}}^+(C) \left[ \frac{p}{[p]} \right]$  and hence is a flat  $\tilde{\mathbf{B}}_{\text{rig}}^+(C)$ -algebra.

### 3 $\varphi$ -Modules

Let  $L$  and  $C$  be as in the previous section. To shorten notation we will drop the denotation  $(C)$  from the rings introduced there. We recall some definitions and facts from Kedlaya [Ked05]. Let  $a$  be a positive integer.

**Definition 3.1.** A  $\varphi^a$ -module over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  is a finite free  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ -module  $\mathbf{M}$  with a bijective  $\varphi^a$ -semilinear map  $\varphi_{\mathbf{M}} : \mathbf{M} \rightarrow \mathbf{M}$ . The rank of  $\mathbf{M}$  as a  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ -module is denoted  $\text{rk } \mathbf{M}$ . A *morphism of  $\varphi^a$ -modules* is a morphism of the underlying  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ -modules which commutes with the  $\varphi_{\mathbf{M}}$ 's. We denote the set of morphism between two  $\varphi^a$ -modules  $\mathbf{M}$  and  $\mathbf{M}'$  by  $\text{Hom}_{\varphi^a}(\mathbf{M}, \mathbf{M}')$ .

**Example 3.2.** Let  $c, d \in \mathbb{Z}$  with  $d > 0$  and  $(c, d) = 1$ . Define the  $\varphi^a$ -module  $\mathbf{M}(c, d)$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  as  $\mathbf{M}(c, d) = \bigoplus_{i=1}^d \tilde{\mathbf{B}}_{\text{rig}}^\dagger \mathbf{e}_i$  equipped with

$$\varphi_{\mathbf{M}}(\mathbf{e}_1) = \mathbf{e}_2, \quad \dots, \quad \varphi_{\mathbf{M}}(\mathbf{e}_{d-1}) = \mathbf{e}_d, \quad \varphi_{\mathbf{M}}(\mathbf{e}_d) = p^c \mathbf{e}_1. \quad (3.1)$$

By abuse of notation we also write  $\mathbf{M}(nc, nd) := \mathbf{M}(c, d)^{\oplus n}$  for  $n \in \mathbb{N}_{>0}$ .

The category of  $\varphi^a$ -modules over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  is a rigid additive tensor category with unit object  $\mathbf{M}(0, 1)$ . For a positive integer  $b$  there is a *restriction of Frobenius functor*  $[b]_*$  from  $\varphi^a$ -modules over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  to  $\varphi^{ab}$ -modules over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  sending  $(\mathbf{M}, \varphi_{\mathbf{M}})$  to  $(\mathbf{M}, \varphi_{\mathbf{M}}^b)$ . Kedlaya proved the following structure theorem.

**Theorem 3.3.** ([Ked05, Theorem 4.5.7]) *Any  $\varphi^a$ -module  $\mathbf{M}$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  is isomorphic to a direct sum of  $\varphi^a$ -modules  $\mathbf{M}(c_i, d_i)$  for uniquely determined pairs  $(c_i, d_i)$  up to permutation with  $\gcd(c_i, d_i) = 1$ . It satisfies  $\wedge^d \mathbf{M} \cong \mathbf{M}(c, 1)$  where  $c = \sum_i c_i$  and  $d = \text{rk } \mathbf{M} = \sum_i d_i$ .*

**Definition 3.4.** If  $\det \mathbf{M} := \wedge^{\text{rk } \mathbf{M}} \mathbf{M} \cong \mathbf{M}(c, 1)$  we define the *degree* and the *weight* of  $\mathbf{M}$  as  $\text{deg } \mathbf{M} := c$  and  $\text{wt } \mathbf{M} := \frac{\text{deg } \mathbf{M}}{\text{rk } \mathbf{M}}$ .

**Proposition 3.5.** ([Ked05, Lemma 3.4.9]) *Every  $\varphi^a$ -submodule  $\mathbf{M}' \subset \mathbf{M}(c, d)^{\oplus n}$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  satisfies  $\text{wt } \mathbf{M}' \geq \text{wt } \mathbf{M}(c, d)^{\oplus n} = \frac{c}{d}$ .*

Let us record some facts about the  $\mathbf{M}(c, d)$ .

**Lemma 3.6.** ([Ked05, Lemmas 4.1.2 and 3.2.4]) *The  $\varphi^a$ -modules  $\mathbf{M}(c, d)$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  satisfy*

$$(a) \mathbf{M}(c, d) \otimes \mathbf{M}(c', d') \cong \mathbf{M}(cd' + c'd, dd'),$$

$$(b) \mathbf{M}(c, d)^\vee \cong \mathbf{M}(-c, d) \text{ and}$$

$$(c) [d]_* \mathbf{M}(c, d) \cong \mathbf{M}(c, 1)^{\oplus d}.$$

**Definition 3.7.** For a  $\varphi^a$ -module  $\mathbf{M}$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  we define the set of  $\varphi^a$ -invariants as

$$H_{\varphi^a}^0(\mathbf{M}) := \{x \in \mathbf{M} : \varphi_{\mathbf{M}}(x) = x\}.$$

It is a vector space over  $H_{\varphi^a}^0(\mathbf{M}(0, 1)) = \mathbb{Q}_{p^a} := W(\mathbb{F}_{p^a})[p^{-1}]$ ; use [Ked05, Proposition 3.3.4]. We have  $\text{Hom}_{\varphi^a}(\mathbf{M}, \mathbf{M}') = H_{\varphi^a}^0(\mathbf{M}^\vee \otimes \mathbf{M}')$  for  $\varphi^a$ -modules  $\mathbf{M}$  and  $\mathbf{M}'$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ .

**Proposition 3.8.** *If  $a \geq c > 0$  then the  $\varphi^a$ -module  $\mathbf{M}(-c, 1)$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  satisfies*

$$H_{\varphi^a}^0(\mathbf{M}(-c, 1)) = \left\{ \sum_{\nu \in \mathbb{Z}} p^{c\nu} \sum_{j=0}^{c-1} p^j \varphi^{-a\nu}([x_j]) : x_0, \dots, x_{c-1} \in \tilde{\mathbf{E}}, v_{\mathbf{E}}(x_j) > 0 \right\}.$$

*Proof.* One easily verifies that the series on the right converges (even in  $\tilde{\mathbf{B}}^{[0, r]}$  for all  $r > 0$ ) and is a  $\varphi^a$ -invariant of  $\mathbf{M}(-c, 1)$ .

Conversely let  $x \in \tilde{\mathbf{B}}_{\text{rig}}^\dagger$  satisfy  $x = p^{-c} \varphi^a(x)$ . Then  $x \in \tilde{\mathbf{B}}_{\text{rig}}^+$  by [Ber04, Proposition I.4.1] and hence possesses an expansion  $x = \sum_{j=-\infty}^{\infty} p^j [x_j]$  with uniquely determined  $x_j \in \tilde{\mathbf{E}}$  satisfying  $v_{\mathbf{E}}(x_j) > 0$  by [Fou05, Lemme 3.9.17 and Corollaire 3.9.9]. The uniqueness implies  $x_j^{p^a} = x_{j-c}$  and the proposition follows.  $\square$

Let us make a remark on radii of convergence. Let  $\mathbf{M}^{[0, r]}$  be a free  $\tilde{\mathbf{B}}^{[0, r]}$ -module and for  $0 < s \leq r$  let  $\mathbf{M}^{[0, s]} := \mathbf{M}^{[0, r]} \otimes_{\tilde{\mathbf{B}}^{[0, r]}, \iota} \tilde{\mathbf{B}}^{[0, s]}$  be obtained by base change via the natural inclusion  $\iota : \tilde{\mathbf{B}}^{[0, r]} \hookrightarrow \tilde{\mathbf{B}}^{[0, s]}$ . Let further

$$\varphi_{\mathbf{M}}^{[0, rp^{-a}]} : \mathbf{M}^{[0, r]} \otimes_{\tilde{\mathbf{B}}^{[0, r]}, \varphi^a} \tilde{\mathbf{B}}^{[0, rp^{-a}]} \xrightarrow{\sim} \mathbf{M}^{[0, rp^{-a}]}$$

be an isomorphism of  $\tilde{\mathbf{B}}^{[0, rp^{-a}]}$ -modules. We say that a  $\varphi^a$ -module  $(\mathbf{M}, \varphi_{\mathbf{M}})$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  is *represented* by the pair  $(\mathbf{M}^{[0, r]}, \varphi_{\mathbf{M}}^{[0, rp^{-a}]})$  if  $(\mathbf{M}, \varphi_{\mathbf{M}}) = (\mathbf{M}^{[0, r]} \otimes_{\tilde{\mathbf{B}}^{[0, r]}} \tilde{\mathbf{B}}_{\text{rig}}^\dagger, \varphi_{\mathbf{M}}^{[0, rp^{-a}]} \otimes \text{id})$ .

**Proposition 3.9.** *If  $\mathbf{M}$  is represented by  $(\mathbf{M}^{[0, r]}, \varphi_{\mathbf{M}}^{[0, rp^{-a}]})$  then*

$$H_{\varphi^a}^0(\mathbf{M}) = \{x \in \mathbf{M}^{[0, r]} : \varphi_{\mathbf{M}}^{[0, rp^{-a}]}(x \otimes_{\varphi^a} 1) = x \otimes_{\iota} 1\}.$$

*Proof.* The inclusion “ $\supset$ ” follows from the inclusion  $\mathbf{M}^{[0, r]} \subset \mathbf{M}$ .

To prove the opposite inclusion “ $\subset$ ” let  $x \in H_{\varphi^a}^0(\mathbf{M})$ . Since  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger = \bigcup_{s>0} \tilde{\mathbf{B}}^{[0, s]}$  there exists a  $0 < s \leq r$  with  $x \in \mathbf{M}^{[0, s]}$ . Choose a  $\tilde{\mathbf{B}}^{[0, r]}$ -basis of  $\mathbf{M}^{[0, r]}$  and write  $x$  with respect to this basis as a vector  $v = (v_1, \dots, v_n)^T \in (\tilde{\mathbf{B}}^{[0, s]})^{\oplus n}$ . Let  $A \in \text{GL}_n(\tilde{\mathbf{B}}^{[0, rp^{-a}]})$  be the matrix by which the isomorphism  $\varphi_{\mathbf{M}}^{[0, rp^{-a}]}$  acts on this basis. The equation  $x = \varphi_{\mathbf{M}}(x)$  translates into  $A^{-1}v = (\varphi^a(v_1), \dots, \varphi^a(v_n))^T$ . Thus if  $s \leq rp^{-a}$  we find  $\varphi^a(v_i) \in \tilde{\mathbf{B}}^{[0, s]}$ , whence  $v_i \in \tilde{\mathbf{B}}^{[0, sp^a]}$ . Continuing in this way we see that in fact  $v_i \in \tilde{\mathbf{B}}^{[0, r]}$ , that is  $x \in \mathbf{M}^{[0, r]}$ .  $\square$

**Proposition 3.10.** *Let  $c, d > 0$  and let the  $\varphi$ -module  $\mathbf{M} = \mathbf{M}(-c, d)$  over  $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger}(C)$  be represented by  $\mathbf{M}^{[0,1]} = \bigoplus_{i=1}^d \tilde{\mathbf{B}}^{[0,1]} \mathbf{e}_i$  and  $\varphi^{[0,p^{-1}]}$  as in (3.1). Then  $\theta : \tilde{\mathbf{B}}^{[0,1]}(C) \rightarrow C$  induces the following exact sequence of  $\mathbb{Q}_p$ -vector spaces*

$$0 \longrightarrow H_{\varphi}^0(\mathbf{M}(d-c, d)) \xrightarrow{T_{-c,d}} H_{\varphi}^0(\mathbf{M}(-c, d)) \xrightarrow{\theta_{\mathbf{M}}} \bigoplus_{i=1}^d C \mathbf{e}_i \longrightarrow 0,$$

$$\sum_{i=1}^d x_i \mathbf{e}_i \longmapsto \sum_{i=1}^d p^{i-1} t x_i \mathbf{e}_i, \quad \sum_{i=1}^d y_i \mathbf{e}_i \longmapsto \sum_{i=1}^d \theta(y_i) \mathbf{e}_i.$$

In particular  $\theta_{\mathbf{M}}$  is an isomorphism for  $0 < c < d$ .

*Proof.* The elements of  $H_{\varphi}^0(\mathbf{M}(-c, d))$  are of the form  $\sum_{i=1}^d \varphi^{i-1}(y) \mathbf{e}_i$  for  $y = p^{-c} \varphi^d(y)$  and  $y \in \tilde{\mathbf{B}}_{\text{rig}}^{\dagger} \subset \tilde{\mathbf{B}}_{\text{max}}^{\dagger} \subset \tilde{\mathbf{B}}_{\text{rig}}^{\dagger}$  by [Ber04, Proposition I.4.1]. The map  $\theta_{\mathbf{M}}$  sends this element to  $\sum_{i=1}^d \theta(\varphi^{i-1}(y)) \mathbf{e}_i$ . Since  $\{\varphi^{i-1} : 1 \leq i \leq d\} = \text{Gal}(\mathbb{Q}_{p^d}/\mathbb{Q}_p)$ , the sequence is exact on the right by [Col02, Lemme 8.16]. Since  $\varphi(t) = pt$  and  $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger}$  is an integral domain the sequence is well defined and exact on the left. Finally, the exactness in the middle follows from  $t \cdot \tilde{\mathbf{B}}_{\text{max}}^{\dagger} = \{y \in \tilde{\mathbf{B}}_{\text{max}}^{\dagger} : \theta(\varphi^n(y)) = 0 \text{ for all } n \in \mathbf{N}_0\}$ ; see [Col02, Proposition 8.10]. If  $0 < c < d$  then  $H_{\varphi}^0(\mathbf{M}(d-c, d)) = (0)$  by [Ked05, Proposition 4.1.3] and  $\theta_{\mathbf{M}}$  is an isomorphism.  $\square$

**Corollary 3.11.** *If  $A : \mathbf{M} \rightarrow \mathbf{M}(c, d)$  is a homomorphism of  $\varphi$ -modules over  $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger}$  represented by  $A^{[0,1]} : \mathbf{M}^{[0,1]} \rightarrow (\tilde{\mathbf{B}}^{[0,1]})^{\oplus d}$  with  $\theta_{\mathbf{M}(c,d)} \circ A^{[0,1]}(\mathbf{M}^{[0,1]}) = (0)$  then  $A$  factors as  $A = T \circ B$  for a morphism  $B : \mathbf{M} \rightarrow \mathbf{M}(c+d, d)$  where  $T = T_{c,d} : \mathbf{M}(c+d, d) \rightarrow \mathbf{M}(c, d)$  is induced by multiplication with  $t$  as in Proposition 3.10.*

*Proof.* By Theorem 3.3 and Lemma 3.6 we write  $\mathbf{M} \cong \sum_{i=1}^r \mathbf{M}(c_i, d_i)$  and

$$\text{Hom}_{\varphi}(\mathbf{M}, \mathbf{M}(c, d)) \cong \bigoplus_{i=1}^r H_{\varphi}^0(\mathbf{M}(cd_i - c_i d, dd_i)).$$

Then the assertion follows from Proposition 3.10.  $\square$

**Proposition 3.12.** *The  $\varphi^a$ -modules  $\mathbf{M}(c, d)$  over  $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger}$  satisfy*

$$(a) \dim_{\mathbb{Q}_{p^a}} H_{\varphi^a}^0(\mathbf{M}(c, d)) = \begin{cases} 0 & \text{if } c/d > 0 \\ 1 & \text{if } c = 0, d = 1 \\ \infty & \text{if } c/d < 0 \end{cases}$$

$$(b) \dim_{\mathbb{Q}_{p^a}} \text{Hom}_{\varphi^a}(\mathbf{M}(c, d), \mathbf{M}(c', d')) = \begin{cases} 0 & \text{if } c/d < c'/d', \\ d^2 & \text{if } c = c', d = d', \\ \infty & \text{if } c/d > c'/d'. \end{cases}$$

(c)  $\text{End}_{\varphi^a}(\mathbf{M}(c, d))$  is a central division algebra over  $\mathbb{Q}_{p^a}$  of dimension  $d^2$  and Hasse invariant  $\frac{c}{d}$ .

*Proof.* (a) The assertions for  $c/d > 0$  and  $c = 0, d = 1$  were proved by Kedlaya [Ked05, Propositions 4.1.3 and 3.3.4]. The case  $c/d < 0$  follows from Proposition 3.10 since  $\dim_{\mathbb{Q}_{p^a}} C = \infty$ . Note that Proposition 3.10 was proved only for  $a = 1$  but the surjectivity of  $\theta_{\mathbf{M}}$ , which only relies on [Col02, Lemme 8.16], holds also for  $a > 1$ .

(b) follows from (a) and

$$\mathrm{Hom}_{\varphi^a}(\mathbf{M}(c, d), \mathbf{M}(c', d')) = \mathrm{H}_{\varphi^a}^0(\mathbf{M}(-c, d) \otimes \mathbf{M}(c', d')) \cong \mathrm{H}_{\varphi^a}^0(\mathbf{M}(c'd - cd', dd')).$$

(c) was proved in [Ked05, Remark 4.1.5]. The Hasse invariant can be computed as in [HP04, Proposition 8.6].  $\square$

## 4 Constructing $\varphi$ -Modules from Filtered Isocrystals

We keep the notation from Section 2. In the situation where the minimal and maximal Hodge-Tate weights differ by at most 1 we will associate with any analytic point  $\mu \in \check{\mathcal{F}}^{\mathrm{an}}$  a  $\varphi$ -module  $\mathbf{M}_\mu$  over  $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C)$  where  $C$  is the completion of an algebraic closure of  $\mathcal{H}(\mu)$ . In case  $\mathcal{H}(\mu)/\check{E}$  finite this construction parallels Berger's construction [Ber09, §II] that works for arbitrary Hodge-Tate weights and even produces an “ $\mathcal{H}(\mu)$ -rational” version of  $\mathbf{M}_\mu$ . We begin with the following

**Proposition 4.1.** *Let  $\mathbf{N}$  be a  $\varphi$ -module over  $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$  represented by a free  $\tilde{\mathbf{B}}^{[0,1]}$ -module  $\mathbf{N}^{[0,1]}$  and an isomorphism  $\varphi_{\mathbf{N}}^{[0,p^{-1}]} : \mathbf{N}^{[0,1]} \otimes_{\tilde{\mathbf{B}}^{[0,1],\varphi}} \tilde{\mathbf{B}}^{[0,p^{-1}]} \xrightarrow{\sim} \mathbf{N}^{[0,p^{-1}]}$ . Consider the morphism  $\theta_{\mathbf{N}} = \mathrm{id}_{\mathbf{N}} \otimes \theta : \mathbf{N}^{[0,1]} \rightarrow \mathbf{N}^{[0,1]} \otimes_{\tilde{\mathbf{B}}^{[0,1],\theta}} C =: W_{\mathbf{N}}$  and let  $W_{\mathbf{M}}$  be a  $C$ -subspace of  $W_{\mathbf{N}}$ . Then there exists a uniquely determined  $\varphi$ -submodule  $\mathbf{M} \subset \mathbf{N}$  over  $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$  with  $t\mathbf{N} \subset \mathbf{M}$  which is represented by a  $\tilde{\mathbf{B}}^{[0,1]}$ -submodule  $t\mathbf{N}^{[0,1]} \subset \mathbf{M}^{[0,1]} \subset \mathbf{N}^{[0,1]}$  such that  $\theta_{\mathbf{N}}(\mathbf{M}^{[0,1]}) = W_{\mathbf{M}}$ .*

*Proof.* If  $I$  is a closed subinterval of  $(0, 1]$  set  $\mathbf{N}^I := \mathbf{N}^{[0,1]} \otimes_{\tilde{\mathbf{B}}^{[0,1]}} \tilde{\mathbf{B}}^I$ . Also if  $I = [s, r]$  we let  $pI := [ps, pr]$ . The isomorphism  $\varphi_{\mathbf{N}}^{[0,p^{-1}]}$  induces an isomorphism  $\varphi_{\mathbf{N}}^I : \mathbf{N}^{pI} \otimes_{\tilde{\mathbf{B}}^{pI,\varphi}} \tilde{\mathbf{B}}^I \xrightarrow{\sim} \mathbf{N}^I$ . Let  $I_n := [p^{-n-1}, p^{-n}]$  for  $n \in \mathbb{N}_0$ . We define  $\mathbf{M}^{I_0}$  as the preimage of  $W_{\mathbf{M}}$  in  $\mathbf{N}^{I_0}$  under the canonical morphism  $\mathbf{N}^{I_0} \rightarrow \mathbf{N}^{I_0} \otimes_{\tilde{\mathbf{B}}^{I_0,\theta}} C = W_{\mathbf{N}}$  and we let  $\mathbf{M}^{I_0}$  be the intersection of  $\mathbf{M}^{I_0}$  with  $\varphi_{\mathbf{N}}^{[p^{-1},p^{-1}]}(\mathbf{M}^{I_0} \otimes_{\tilde{\mathbf{B}}^{I_0,\varphi}} \tilde{\mathbf{B}}^{[p^{-1},p^{-1}]})$  inside  $\mathbf{N}^{[p^{-1},p^{-1}]}$ , that is  $\mathbf{M}^{I_0}$  equals

$$\ker \left( \mathbf{M}^{I_0} \hookrightarrow \mathbf{N}^{[p^{-1},p^{-1}]} \longrightarrow \mathbf{N}^{[p^{-1},p^{-1}]} / \varphi_{\mathbf{N}}^{[p^{-1},p^{-1}]}(\mathbf{M}^{I_0} \otimes_{\tilde{\mathbf{B}}^{I_0,\varphi}} \tilde{\mathbf{B}}^{[p^{-1},p^{-1}]}) \right).$$

Since  $\tilde{\mathbf{B}}^{I_0}$  is a principal ideal domain by [Ked05, Proposition 2.6.8] we see that  $\mathbf{M}^{I_0}$  is a free  $\tilde{\mathbf{B}}^{I_0}$ -submodule of  $\mathbf{N}^{I_0}$  of full rank. For  $n \geq 1$  we define  $\mathbf{M}^{I_n}$  as the image of  $\mathbf{M}^{I_0} \otimes_{\tilde{\mathbf{B}}^{I_0,\varphi^n}} \tilde{\mathbf{B}}^{I_n}$  under the isomorphism  $\varphi_{\mathbf{N}}^{I_n} \circ \dots \circ \varphi_{\mathbf{N}}^{I_1} : \mathbf{N}^{I_0} \otimes_{\tilde{\mathbf{B}}^{I_0,\varphi^n}} \tilde{\mathbf{B}}^{I_n} \xrightarrow{\sim} \mathbf{N}^{I_n}$ . In the terminology of Kedlaya [Ked05, §2.8] the collection  $\mathbf{M}^{I_n}$  for  $n \in \mathbb{N}_0$  defines a *vector bundle over  $\tilde{\mathbf{B}}^{[0,1]}$*  which by [Ked05, Theorem 2.8.4] corresponds to a free  $\tilde{\mathbf{B}}^{[0,1]}$ -submodule  $\mathbf{M}^{[0,1]}$  of  $\mathbf{N}^{[0,1]}$ . By construction  $t\mathbf{N}^{[0,1]} \subset \mathbf{M}^{[0,1]}$  and  $\varphi_{\mathbf{N}}^{[0,p^{-1}]}$  restricts to an isomorphism on  $\mathbf{M}^{[0,1]}$ . This makes  $\mathbf{M} := \mathbf{M}^{[0,1]} \otimes_{\tilde{\mathbf{B}}^{[0,1]}} \tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$  into a  $\varphi$ -module with the desired properties. Clearly  $\mathbf{M}^{[0,1]}$  is uniquely determined by the subspace  $W_{\mathbf{M}} \subset W_{\mathbf{N}}$  and the requirements that  $t\mathbf{N}^{[0,1]} \subset \mathbf{M}^{[0,1]} \subset \mathbf{N}^{[0,1]}$  and  $\varphi_{\mathbf{N}}^{I_n}(\mathbf{M}^{I_{n-1}} \otimes_{\tilde{\mathbf{B}}^{I_{n-1},\varphi}} \tilde{\mathbf{B}}^{I_n}) = \mathbf{M}^{I_n}$ .  $\square$

Now assume that  $L$  is a (not necessarily finite) extension of  $K_0$  and let  $h$  be an integer. Let  $(D, \varphi_D)$  be an  $F$ -isocrystal over  $\mathbb{F}_p^{\mathrm{alg}}$  and let  $\mathrm{Fil}^h D_L$  be an  $L$ -subspace of  $D_L = D \otimes_{K_0} L$ . Let  $\mathrm{Fil}^{h-1} D_L = D_L$  and  $\mathrm{Fil}^{h+1} D_L = (0)$ . Then  $\underline{D} = (D, \varphi_D, \mathrm{Fil}^\bullet D_L)$  is a filtered isocrystal over  $L$  with Hodge-Tate weights  $-h$  and  $-h+1$ .

**Definition 4.2.** We set  $\mathbf{D}^{[0,1]} := D \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,1]}$  and  $\mathbf{D}(\underline{D}) := (\mathbf{D}, \varphi_{\mathbf{D}}) := (D, \varphi_D) \otimes_{K_0} \widetilde{\mathbf{B}}_{\text{rig}}^\dagger$ . Multiplication with  $t^h$  defines natural isomorphisms  $\text{id}_D \otimes t^h : t^{-h}\mathbf{D}^{[0,1]} \xrightarrow{\sim} D \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,1]}$  and  $t^{-h}\mathbf{D}(\underline{D}) \xrightarrow{\sim} (D, p^{-h}\varphi_D) \otimes_{K_0} \widetilde{\mathbf{B}}_{\text{rig}}^\dagger$ . We let  $\mathbf{M}(\underline{D})$  be the  $\varphi$ -submodule of  $t^{-h}\mathbf{D}$  over  $\widetilde{\mathbf{B}}_{\text{rig}}^\dagger$  represented by the  $\widetilde{\mathbf{B}}^{[0,1]}$ -submodule  $\mathbf{M}^{[0,1]}$  of  $t^{-h}\mathbf{D}^{[0,1]}$  with  $\theta_{t^{-h}\mathbf{D}}(\mathbf{M}^{[0,1]}) = (\text{Fil}^h D_L) \otimes_L C$  under the map  $\theta_{t^{-h}\mathbf{D}} : t^{-h}\mathbf{D}^{[0,1]} \rightarrow t^{-h}\mathbf{D}^{[0,1]} \otimes_{\widetilde{\mathbf{B}}^{[0,1],\theta}} C = D_C$ , whose existence was established in Proposition 4.1.

The following lemma is immediate from the definition.

**Lemma 4.3.** For all integers  $n$  consider the Tate object  $\mathbf{1}(n) := (D = K_0, \varphi_D = p^n, K_0 = \text{Fil}^n \supset \text{Fil}^{n+1} = (0))$ . Then the canonical isomorphism  $\text{id}_D \otimes t^n : \mathbf{D}(\underline{D} \otimes \mathbf{1}(n)) \xrightarrow{\sim} t^n \mathbf{D}(\underline{D})$  induces an isomorphism  $\mathbf{M}(\underline{D} \otimes \mathbf{1}(n)) \cong \mathbf{M}(\underline{D})$ .  $\square$

The construction is compatible with dualizing in the following sense.

**Proposition 4.4.**  $\mathbf{M}(\underline{D}^\vee) \cong \mathbf{M}(\underline{D})^\vee := \text{Hom}_\varphi(\mathbf{M}(\underline{D}), \mathbf{M}(0, 1))$

*Proof.* If  $\underline{D}$  is a filtered isocrystal with  $\text{Fil}^{h-1} D_L = D_L \supset \text{Fil}^h D_L \supset \text{Fil}^{h+1} D_L = (0)$  then  $\underline{D}^\vee$  has  $\text{Fil}^{-h} D_L^\vee = D_L^\vee \supset \text{Fil}^{-h+1} D_L^\vee \supset \text{Fil}^{-h+2} D_L^\vee = (0)$  because  $\text{Fil}^i D_L^\vee := \{\lambda \in D_L^\vee : \lambda(\text{Fil}^{1-i} D_L) = (0)\}$  for all  $i$ . Thus the canonical pairing  $D \times D^\vee \rightarrow K_0$  induces a pairing  $\psi$  as follows

$$\begin{array}{ccccc}
\mathbf{M}(\underline{D})^{[0,1]} & \times & \mathbf{M}(\underline{D}^\vee)^{[0,1]} & & \\
\downarrow & & \downarrow & & \\
t^{-h}\mathbf{D}(\underline{D})^{[0,1]} & \times & t^{h-1}\mathbf{D}(\underline{D}^\vee)^{[0,1]} & \xrightarrow{\psi} & \mathbf{M}(-1, 1)^{[0,1]} \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
(D, p^{-h}\varphi_D) \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,1]} & \times & (D^\vee, p^{h-1}\varphi_{D^\vee}) \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,1]} & \longrightarrow & (K_0, p^{-1}\varphi) \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,1]} \\
\downarrow \theta_{t^{-h}\mathbf{D}(\underline{D})} & & \downarrow \theta_{t^{h-1}\mathbf{D}(\underline{D}^\vee)} & & \downarrow \theta_{\mathbf{M}(-1,1)} \\
D_L & \times & D_L^\vee & \longrightarrow & L
\end{array}$$

Since  $\theta_{t^{-h}\mathbf{D}(\underline{D})}(\mathbf{M}(\underline{D})^{[0,1]}) = \text{Fil}^h D_C$  and  $\theta_{t^{h-1}\mathbf{D}(\underline{D}^\vee)}(\mathbf{M}(\underline{D}^\vee)^{[0,1]}) = \text{Fil}^{-h+1} D_C^\vee$  we obtain  $\theta_{\mathbf{M}(-1,1)} \circ \psi(\mathbf{M}(\underline{D})^{[0,1]} \otimes_{\widetilde{\mathbf{B}}^{[0,1]}} \mathbf{M}(\underline{D}^\vee)^{[0,1]}) = (0)$ . By Corollary 3.11 the pairing  $\mathbf{M}(\underline{D}) \otimes_{\widetilde{\mathbf{B}}_{\text{rig}}^\dagger} \mathbf{M}(\underline{D}^\vee) \rightarrow \mathbf{M}(-1, 1)$  factors through  $\mathbf{M}(0, 1)$ . This implies  $\mathbf{M}(\underline{D}^\vee) \hookrightarrow \mathbf{M}(\underline{D})^\vee$ . Since both  $\varphi$ -modules have rank equal to  $\dim D$  and by Theorem 4.5 below their degree satisfies

$$\deg \mathbf{M}(\underline{D}^\vee) = t_N(\underline{D}^\vee) - t_H(\underline{D}^\vee) = -t_N(\underline{D}) + t_H(\underline{D}) = -\deg \mathbf{M}(\underline{D}) = \deg \mathbf{M}(\underline{D})^\vee,$$

[Ked05, Lemma 3.4.2] yields  $\mathbf{M}(\underline{D}^\vee) \xrightarrow{\sim} \mathbf{M}(\underline{D})^\vee$ .  $\square$

Unfortunately we do not know how to construct  $\mathbf{M}(\underline{D})$  for filtered isocrystals  $\underline{D}$  with Hodge-Tate weights other than  $-h$  and  $-h + 1$ . Therefore we cannot make  $\underline{D} \mapsto \mathbf{M}(\underline{D})$  into a tensor functor and we do not follow Berger's argument [Ber09, Théorème IV.2.1] to prove the next theorem.

**Theorem 4.5.**  $\deg \mathbf{M}(\underline{D}) = t_N(\underline{D}) - t_H(\underline{D})$ .

*Proof.* By the Dieudonné-Manin classification [Man63] there exists an  $F$ -isocrystal  $(D', \varphi_{D'})$  over  $\mathbb{F}_p^{\text{alg}}$  of rank one with  $\varphi_{D'} = p^{t_N(D)} \cdot \varphi$  which is isomorphic to  $\det(D, \varphi_D)$ . So by construction  $\deg \mathbf{D}(D) = t_N(D)$ . Moreover,

$$\begin{aligned} t_H(D) &= (h-1) \cdot \dim_L(D_L/\text{Fil}^h D_L) + h \cdot \dim_L \text{Fil}^h D_L \\ &= h \cdot \dim_L D_L - \dim_C(D_L/\text{Fil}^h D_L) \otimes_L C \\ &= h \cdot \dim_{K_0} D - \dim_C(t^{-h} \mathbf{D}^{[0,1]}/\mathbf{M}^{[0,1]}) \otimes_{\tilde{\mathbf{B}}^{[0,1],\theta}} C. \end{aligned}$$

The following lemma implies that

$$\begin{aligned} \deg t^n \mathbf{D}(D) - \deg t^{n-1} \mathbf{D}(D) &= \dim_{K_0} D \quad \text{for all } n \text{ and} \\ \deg \mathbf{M}(D) - \deg t^{-h} \mathbf{D}(D) &= \dim_C(t^{-h} \mathbf{D}^{[0,1]}/\mathbf{M}^{[0,1]}) \otimes_{\tilde{\mathbf{B}}^{[0,1],\theta}} C \end{aligned}$$

and this proves the theorem.  $\square$

**Lemma 4.6.** *Let  $\mathbf{M}_1$  and  $\mathbf{M}_2$  be  $\varphi$ -modules over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  represented by  $\tilde{\mathbf{B}}^{[0,1]}$ -modules  $\mathbf{M}_i^{[0,1]}$ . Assume that  $\mathbf{M}_1^{[0,1]} \supset \mathbf{M}_2^{[0,1]} \supset t\mathbf{M}_1^{[0,1]}$ . Then*

$$\deg \mathbf{M}_2 - \deg \mathbf{M}_1 = \dim_C(\mathbf{M}_1^{[0,1]}/\mathbf{M}_2^{[0,1]}) \otimes_{\tilde{\mathbf{B}}^{[0,1],\theta}} C.$$

*Proof.* Clearly the equality holds for  $\mathbf{M}_2 = t\mathbf{M}_1 \cong \mathbf{M}_1 \otimes \mathbf{M}(1, 1)$  since  $\deg t\mathbf{M}_1 - \deg \mathbf{M}_1 = \text{rk } \mathbf{M}_1$ . We claim that it suffices to prove the inequality

$$\deg \mathbf{M}_2 - \deg \mathbf{M}_1 \geq \dim_C W \quad (4.1)$$

where we abbreviate  $W := (\mathbf{M}_1^{[0,1]}/\mathbf{M}_2^{[0,1]}) \otimes_{\tilde{\mathbf{B}}^{[0,1],\theta}} C$ . Indeed we apply the inequality to the two inclusions  $\mathbf{M}_1 \supset \mathbf{M}_2 \supset t\mathbf{M}_1$  and  $\mathbf{M}_2 \supset t\mathbf{M}_1 \supset t\mathbf{M}_2$  and conclude using the exact sequence of  $\tilde{\mathbf{B}}^{[0,1]}$ -modules

$$0 \longrightarrow \mathbf{M}_2^{[0,1]}/t\mathbf{M}_1^{[0,1]} \longrightarrow \mathbf{M}_1^{[0,1]}/t\mathbf{M}_1^{[0,1]} \longrightarrow \mathbf{M}_1^{[0,1]}/\mathbf{M}_2^{[0,1]} \longrightarrow 0.$$

To prove the inequality (4.1) we argue by induction on  $\dim_C W$ . Let  $\dim_C W = 1$ . Since  $\det \mathbf{M}_1 \supset \det \mathbf{M}_2$  we know that  $\deg \mathbf{M}_2 \geq \deg \mathbf{M}_1$  from Proposition 3.12. If we had  $\deg \mathbf{M}_2 = \deg \mathbf{M}_1$  then  $\mathbf{M}_2 = \mathbf{M}_1$  by [Ked05, Lemma 3.4.2]. So  $\deg \mathbf{M}_2 - \deg \mathbf{M}_1 \geq 1 = \dim_C W$  as desired.

Let now  $\dim_C W > 1$  and choose a  $C$ -subspace  $W'$  of dimension 1 of  $W$ . By Proposition 4.1 there is a unique  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ -submodule  $\mathbf{M}_2 \subset \mathbf{M}_3 \subset \mathbf{M}_1$  corresponding to  $W'$ . By induction  $\deg \mathbf{M}_3 - \deg \mathbf{M}_1 \geq \dim_C(W/W')$  and  $\deg \mathbf{M}_2 - \deg \mathbf{M}_3 \geq \dim_C W'$  proving the lemma.  $\square$

**Remark 4.7.** As J. Pottharst pointed out to us, one could nevertheless follow [Ber09, Théorème IV.2.1] to prove Theorem 4.5 by working in the exact tensor category  $\underline{\text{MP}}_C^\varphi$  of triples  $(D, \varphi_D, \mathcal{L})$  with  $(D, \varphi_D)$  an  $F$ -isocrystal over  $\mathbb{F}_p^{\text{alg}}$  and  $\mathcal{L}$  a  $\mathbf{B}_{\text{dR}}^+$ -lattice inside  $D \otimes_{K_0} \mathbf{B}_{\text{dR}}$ , where  $\mathbf{B}_{\text{dR}}^+ = \varprojlim \tilde{\mathbf{B}}^+ / (\ker \theta)^n$ ,  $\mathbf{B}_{\text{dR}} = \mathbf{B}_{\text{dR}}^+[t^{-1}]$ , and  $\theta : \mathbf{B}_{\text{dR}}^+ / t\mathbf{B}_{\text{dR}}^+ \xrightarrow{\sim} C$ . Note that to  $\underline{\text{MP}}_C^\varphi$  also the functions  $t_N$  and  $t_H$  extend via  $t_N(D, \varphi_D, \mathcal{L}) := t_N(D, \varphi_D)$  and  $t_H(D, \varphi_D, \mathcal{L}) := n$  if  $\wedge^{\dim D} \mathcal{L} = t^{-n} \wedge^{\dim D} D \otimes_{K_0} \mathbf{B}_{\text{dR}}^+$ . From  $\underline{\text{MP}}_C^\varphi$  there are exact tensor functors  $F_1$  to the category of filtered isocrystals over  $C$  by defining

$$\text{Fil}^i D_C = (t^i \mathcal{L} \cap D \otimes_{K_0} \mathbf{B}_{\text{dR}}^+) / (t^i \mathcal{L} \cap t \cdot D \otimes_{K_0} \mathbf{B}_{\text{dR}}^+) = \text{id}_D \otimes \theta(t^i \mathcal{L} \cap D \otimes_{K_0} \mathbf{B}_{\text{dR}}^+),$$

and  $F_2$  to the category of  $\varphi$ -modules over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  similar to our construction in Proposition 4.1. More precisely  $F_2$  assigns to  $(D, \varphi_D, \mathcal{L})$  the  $\varphi$ -module

$$\begin{aligned} \mathbf{M}(D, \varphi_D, \mathcal{L}) &:= \mathbf{M}(D, \varphi_D, \mathcal{L})^{[0,1]} \otimes_{\tilde{\mathbf{B}}^{[0,1]}} \tilde{\mathbf{B}}_{\text{rig}}^\dagger \quad \text{where} \\ \mathbf{M}(D, \varphi_D, \mathcal{L})^{[0,1]} &:= \{ m \in D \otimes_{K_0} \tilde{\mathbf{B}}^{[0,1]}[t^{-1}] : \varphi_D^i(m \otimes_{\varphi^i} 1) \in \mathcal{L} \text{ for all } i \leq 0 \}. \end{aligned}$$

Restricted to the subcategories where  $t^{-h+1}D \otimes_{K_0} \mathbf{B}_{\text{dR}}^+ \subset \mathcal{L} \subset t^{-h}D \otimes_{K_0} \mathbf{B}_{\text{dR}}^+$ , respectively where the Hodge-Tate weights are  $-h$  and  $-h+1$ , the functor  $F_1$  is an equivalence. The inverse functor  $F_1^{-1}$  assigns to the filtered isocrystal  $\underline{D}$  the  $\mathbf{B}_{\text{dR}}^+$ -lattice  $\mathcal{L}$  such that  $t^h\mathcal{L}$  is the preimage of  $\text{Fil}^h D_C$  under the map  $\text{id}_D \otimes \theta : D \otimes_{K_0} \mathbf{B}_{\text{dR}}^+ \rightarrow D_C$ . Our functor  $\underline{D} \mapsto \mathbf{M}(\underline{D})$  from Definition 4.2 is then the composition  $F_2 \circ F_1^{-1}$ .

**Remark 4.8.** The construction of  $\mathbf{M}(\underline{D})$  is functorial with respect to automorphisms of  $C$ . Therefore  $\mathbf{M}(\underline{D})$  gives rise to a Galois representation

$$\text{Gal}(\mathcal{H}(x)^{\text{alg}}/\mathcal{H}(x)) \longrightarrow \text{GL}\left(\mathbf{H}_\varphi^0(\mathbf{M}(\underline{D}))\right).$$

Of course this representation is of most interest when  $\mathbf{M}(\underline{D}) \cong \mathbf{M}(0, 1)^{\oplus \dim D}$  since then  $\mathbf{H}_\varphi^0(\mathbf{M}(\underline{D})) \cong \mathbb{Q}_p^{\oplus \dim D}$ .

If  $L$  is a discretely valued extension of  $K_0$  one can check that  $\mathbf{M}(\underline{D})$  equals the  $\varphi$ -module over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  constructed by Berger [Ber09, §II]. One of Berger's main theorems is the following criterion.

**Theorem 4.9.** ([Ber09, §IV.2]) *Let  $L$  be a discretely valued extension of  $K_0$ . Then  $\underline{D}$  is weakly admissible if and only if  $\mathbf{M}(\underline{D}) \cong \mathbf{M}(0, 1)^{\oplus \dim D}$ . In this case the crystalline Galois representation associated with  $\underline{D}$  is the representation  $\mathbf{H}_\varphi^0(\mathbf{M}(\underline{D}))$  from Remark 4.8.*

The theorem indicates what the local system on  $\check{\mathcal{F}}_b^a$  could be. We will discuss this in detail in the next sections. Let us first make explicit what happens if  $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0, 1)^{\oplus \dim D}$ .

**Proposition 4.10.** *Assume that  $t_N(\underline{D}) = t_H(\underline{D})$  and that the Hodge-Tate weights of  $\underline{D}$  are  $-h$  and  $-h+1$ . Then  $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0, 1)^{\oplus \dim D}$  if and only if for some (any) integer  $e \geq (\dim D) - 1$  there exists a non-zero  $x \in \mathbf{H}_{\varphi^e}^0(\mathbf{M}(1, 1) \otimes [e]_*(t^{-h}\mathbf{D}))$  with  $\theta_{t^{-h}\mathbf{D}}(\varphi_{t^{-h}\mathbf{D}}^m(x)) \in (\text{Fil}^h D_L) \otimes_L C$  for all  $m = 0, \dots, e-1$ .*

Note that  $t^{-h}\mathbf{D}$  can be represented by  $(D \otimes_{K_0} \tilde{\mathbf{B}}^{[0,r]}, p^{-h}b \cdot \varphi)$  for arbitrarily large  $r > 0$ . Hence by Proposition 3.9 the element  $x$  actually belongs to  $D \otimes_{K_0} \tilde{\mathbf{B}}^{[0,r]}$  and the expression  $\theta_{t^{-h}\mathbf{D}}(\varphi_{t^{-h}\mathbf{D}}^m(x))$  makes sense.

*Proof.* Let  $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0, 1)^{\oplus \dim D}$  and let  $e \geq (\dim D) - 1$  be any integer. By Theorems 4.5 and 3.3 there is a  $\varphi$ -module  $\mathbf{M}(c, d)$  over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  with  $c < 0$  and  $d < \dim D$  which is a summand of  $\mathbf{M}(\underline{D})$ . Thus by Proposition 3.12 there exists a non-zero morphism of  $\varphi$ -modules  $f : \mathbf{M}(-1, e) \rightarrow \mathbf{M}(c, d) \subset \mathbf{M}(\underline{D}) \subset t^{-h}\mathbf{D}$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_e$  be a basis of  $\mathbf{M}(-1, e)$  satisfying (3.1) and let  $x$  be the image of  $\mathbf{e}_1$  in  $t^{-h}\mathbf{D}$  under  $f$ . Then  $\varphi_{t^{-h}\mathbf{D}}^e(x) = p^{-1}x$ , that is  $x \in \mathbf{H}_{\varphi^e}^0(\mathbf{M}(1, 1) \otimes [e]_*(t^{-h}\mathbf{D}))$ . Moreover,  $f(\mathbf{e}_{m+1}) = \varphi_{t^{-h}\mathbf{D}}^m(x)$  in  $t^{-h}\mathbf{D}$  for  $0 \leq m < e$ . Now the fact that the morphism  $f$  factors through  $\mathbf{M}(\underline{D})$  amounts by Definition 4.2 to  $\theta_{t^{-h}\mathbf{D}}(\varphi_{t^{-h}\mathbf{D}}^m(x)) \in (\text{Fil}^h D_L) \otimes_L C$ .

Conversely assume that for some integer  $e \geq (\dim D) - 1$  there exists a non-zero element  $x$  in  $\mathbf{H}_{\varphi^e}^0(\mathbf{M}(1, 1) \otimes [e]_*(t^{-h}\mathbf{D}))$  with  $\theta(\varphi_{t^{-h}\mathbf{D}}^m(x)) \in (\text{Fil}^h D_L) \otimes_L C$  for all  $0 \leq m < e$ . Define

the non-trivial morphism of  $\varphi$ -modules  $f : \mathbf{M}(-1, e) \rightarrow t^{-h}\mathbf{D}$  by  $f(\mathbf{e}_{m+1}) := \varphi_{t^{-h}\mathbf{D}}^m(x)$  for  $0 \leq m < e$ . Since  $\theta_{t^{-h}\mathbf{D}}(\varphi_{t^{-h}\mathbf{D}}^m(x)) \in (Fil^h D_L) \otimes_L C$  the morphism  $f$  factors through  $\mathbf{M}(\underline{D})$  by Definition 4.2. By Proposition 3.12 this implies  $\mathbf{M}(\underline{D}) \not\cong \mathbf{M}(0, 1)^{\oplus \dim D}$ .  $\square$

## 5 The Minuscule Case

We retain the notation from Section 1. In particular  $G$  is a reductive group,  $b \in G(K_0)$ , and  $\{\mu\}$  is a conjugacy class of cocharacters of  $G$ . We make the following assumption on the triple  $(G, b, \{\mu\})$  where  $\nu \in \mathrm{Hom}_{K_0}(\mathbb{G}_m, G) \otimes_{\mathbb{Z}} \mathbb{Q}$  is the slope quasi-cocharacter [RZ96, 1.7] associated with  $b$  :

$$\nu^{\#} = \mu^{\#} \text{ in } (\pi_1(G)_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}) \otimes_{\mathbb{Z}} \mathbb{Q}, \text{ and}$$

$$\text{there is an } n \in \mathbb{Z} \text{ and a faithful } \mathbb{Q}_p\text{-rational representation } V \text{ of } G, \quad (5.1)$$

such that all the weights of  $\{\mu\}$  on  $V$  are  $n - 1$  and  $n$ , and

the isocrystal  $\underline{D}_{b,\mu}(V)$  from (1.4) has Newton slopes in the interval  $[n - 1, n]$ .

The significance of  $\nu^{\#} = \mu^{\#}$  is explained by the following

**Lemma 5.1.** *Let  $\nu$  be the slope quasi-cocharacter in  $\mathrm{Hom}_{K_0}(\mathbb{G}_m, G) \otimes_{\mathbb{Z}} \mathbb{Q}$  associated with  $b$ ; see [RZ96, 1.7], and let  $\rho : G \rightarrow \mathrm{GL}(V)$  be in  $\underline{\mathrm{Rep}}_{\mathbb{Q}_p} G$ . Then under the canonical identification  $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$  we have*

$$\det_V \circ \rho \circ \nu = t_N(\underline{D}_{b,\mu}(V)) \quad \text{and} \quad \det_V \circ \rho \circ \mu = t_H(\underline{D}_{b,\mu}(V)).$$

*In particular, if the images  $\nu^{\#}$  and  $\mu^{\#}$  of  $\nu$  and  $\mu$  in  $(\pi_1(G)_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}) \otimes_{\mathbb{Z}} \mathbb{Q}$  coincide then  $t_N(\underline{D}_{b,\mu}(V)) = t_H(\underline{D}_{b,\mu}(V))$  for all  $V \in \underline{\mathrm{Rep}}_{\mathbb{Q}_p} G$ .*

*Proof.* The statement about  $t_N$  follows from the construction of  $\nu$  in [Kot85, §4.2]. The statement about  $t_H$  is immediate from the definition of  $\underline{D}_{b,\mu}(V)$ . If  $\nu^{\#} = \mu^{\#}$  holds in  $(\pi_1(G)_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}) \otimes_{\mathbb{Z}} \mathbb{Q}$  then  $(\rho \circ \nu)^{\#} = (\rho \circ \mu)^{\#}$  in  $(\pi_1(\mathrm{GL}(V))_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ . Under the last isomorphism we have  $(\rho \circ \nu)^{\#} = \det \circ \rho \circ \nu$  and likewise for  $\mu$ . This proves the lemma.  $\square$

If  $(G, \{\mu\})$  satisfies (5.1) with  $n = 1$  then  $\mu$  is a minuscule coweight by Serre [Ser79, §3]. Furthermore, assume that  $G$  does not possess a proper normal subgroup through which  $\mu$  factors. Then all weights of  $\mu$  in  $V$  form one orbit under the Weyl group, the group  $G$  has no simply connected factor group, and all irreducible components of  $G^{\mathrm{der}}$  are of type  $A_n, B_n, C_n$  or  $D_n$ . Moreover, in case  $V$  is an irreducible representation,  $\dim V$  is even, except possibly when all irreducible factors of  $G^{\mathrm{der}}$  are of type  $A$ .

Assume now that  $(G, b, \{\mu\})$  satisfies condition (5.1) and let  $G' = \mathrm{GL}(V)$ . Then  $\rho : G \hookrightarrow G'$  is a closed embedding. This defines a closed embedding  $\mathcal{F} \hookrightarrow \mathcal{F}' := \mathcal{F}lag(V)$  into the flag variety from (1.2). Here  $\mathcal{F}lag(V)$  is actually the Grassmannian over  $K_0$  of  $c$ -dimensional subspaces of  $V_{K_0} = V \otimes_{\mathbb{Q}_p} K_0$ , where  $c$  is the multiplicity of the weight  $n$  of  $\mu$  on  $V$ . By [RZ96, 1.18] we obtain a closed embedding of  $\check{E}$ -analytic period spaces  $\check{\mathcal{F}}_b^{wa} \hookrightarrow \check{\mathcal{F}}_{\rho(b)}^{wa}, \mu \mapsto \rho \circ \mu$ .

Let  $\mu \in \check{\mathcal{F}}^{\mathrm{an}}$  be an analytic point. Let  $L = \mathcal{H}(\mu)$  and let  $C$  be the completion of an algebraic closure of  $L$ . Let  $\underline{D}_{\mu} := \underline{D}_{b,\mu}(V) := (V_{K_0}, b \cdot \varphi, Fil_{\mu}^{\bullet} D_L)$  be the filtered isocrystal

from (1.4). In particular  $Fil_\mu^{n-1}D_L = D_L$  and  $Fil_\mu^{n+1}D_L = (0)$ . Now we let  $\mathbf{M}_\mu := \mathbf{M}_{b,\mu}(V) := \mathbf{M}(\underline{D}_{b,\mu}(V))$  be the  $\varphi$ -module over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger(C)$  from Definition 4.2. Note that  $\deg \mathbf{M}_\mu = 0$  under our assumption (5.1) by Theorem 4.5 and Lemma 5.1. We define

$$\check{\mathcal{F}}_b^0 := \check{\mathcal{F}}_b^0(V) := \{ \mu \in \check{\mathcal{F}}^{\text{an}} \text{ analytic points} : \mathbf{M}_{b,\mu}(V) \cong \mathbf{M}(0,1)^{\oplus \dim V} \}. \quad (5.2)$$

If  $b' = gb\varphi(g^{-1})$  for some  $g \in G(K_0)$  one easily checks that  $\mu \mapsto g^{-1}\mu g =: \mu'$  maps  $\check{\mathcal{F}}_b^0$  isomorphically onto  $\check{\mathcal{F}}_{b'}^0$ , because  $g$  induces isomorphisms  $\underline{D}_{b',\mu'}(V) \cong \underline{D}_{b,\mu}(V)$  and  $\mathbf{M}_{b',\mu'}(V) \cong \mathbf{M}_{b,\mu}(V)$ . In particular  $\check{\mathcal{F}}_b^0$  is invariant under the group  $J(\mathbb{Q}_p)$  from (1.3).

**Proposition 5.2.** *The set  $\check{\mathcal{F}}_b^0$  is independent of the representation  $V$  of  $G$  with (5.1).*

*Proof.* Let  $\rho : G \hookrightarrow \text{GL}(V)$  and  $\tilde{\rho} : G \hookrightarrow \text{GL}(\tilde{V})$  be two representations of  $G$  satisfying (5.1) for integers  $n$  respectively  $\tilde{n}$ . Let  $\mu \in \check{\mathcal{F}}^{\text{an}}$  and assume  $\mathbf{M}_{b,\mu}(V) \cong \mathbf{M}(0,1)^{\oplus \dim V}$ . We must show that also  $\mathbf{M}_{b,\mu}(\tilde{V}) \cong \mathbf{M}(0,1)^{\oplus \dim \tilde{V}}$ . Since  $V$  is a faithful representation,  $\tilde{V}$  is a direct summand of  $\bigoplus_i V^{\otimes n_i} \otimes_{\mathbb{Q}_p} V^{\vee \otimes m_i}$  for suitable positive integers  $n_i, m_i$ . As in Remark 4.7 consider the object  $(D, \varphi_D, \mathcal{L}) := (V_{K_0}, \rho(b)\varphi, t^{-n}(\text{id}_{V_{K_0}} \otimes \theta)^{-1} Fil^n D_C) \in \underline{\text{MP}}_C^\varphi$  associated with  $\underline{D}_{b,\mu}(V)$  under the inverse  $F_1^{-1}$  of the functor  $F_1$  for Hodge-Tate weights  $-n$  and  $-n+1$ . Set

$$(\tilde{D}, \varphi_{\tilde{D}}, \tilde{\mathcal{L}}) := (\tilde{V}_{K_0}, \tilde{\rho}(b)\varphi, \tilde{V} \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}} \cap \bigoplus_i \mathcal{L}^{\otimes n_i} \otimes_{\mathbf{B}_{\text{dR}}^+} \mathcal{L}^{\vee \otimes m_i})$$

Since  $F_1$  is a tensor functor it assigns to  $(\tilde{D}, \varphi_{\tilde{D}}, \tilde{\mathcal{L}})$  the filtered isocrystal  $\underline{D}_{b,\mu}(\tilde{V}) = (\tilde{D}, \varphi_{\tilde{D}}, Fil_\mu^\bullet \tilde{D}_C)$  over  $C$ . The latter has Hodge-Tate weights  $-\tilde{n}$  and  $-\tilde{n}+1$ , and hence  $t^{-\tilde{n}+1} \tilde{D} \otimes_{K_0} \mathbf{B}_{\text{dR}}^+ \subset \tilde{\mathcal{L}} \subset t^{-\tilde{n}} \tilde{D} \otimes_{K_0} \mathbf{B}_{\text{dR}}^+$  and the inverse  $F_1^{-1}$  of  $F_1$  for Hodge-Tate weights  $-\tilde{n}$  and  $-\tilde{n}+1$  assigns to  $\underline{D}_{b,\mu}(\tilde{V})$  the object  $(\tilde{D}, \varphi_{\tilde{D}}, \tilde{\mathcal{L}})$  of  $\underline{\text{MP}}_C^\varphi$ . Therefore under the tensor functor  $F_2$  we have

$$\mathbf{M}_{b,\mu}(\tilde{V}) = F_2(\tilde{D}, \varphi_{\tilde{D}}, \tilde{\mathcal{L}}) = \tilde{V} \otimes_{\mathbb{Q}_p} \tilde{\mathbf{B}}_{\text{rig}}^\dagger[t^{-1}] \cap \bigoplus_i \mathbf{M}_{b,\mu}(V)^{\otimes n_i} \otimes_{\tilde{\mathbf{B}}_{\text{rig}}^\dagger} \mathbf{M}_{b,\mu}(V)^{\vee \otimes m_i}$$

as  $\varphi$ -modules over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ . By Theorem 3.3 we can write  $\mathbf{M}_{b,\mu}(\tilde{V}) \cong \bigoplus_j \mathbf{M}(\tilde{c}_j, \tilde{d}_j)$ . From  $\mathbf{M}_{b,\mu}(V) \cong \mathbf{M}(0,1)^{\oplus \dim V}$  it follows that also  $\bigoplus_i \mathbf{M}_{b,\mu}(V)^{\otimes n_i} \otimes_{\tilde{\mathbf{B}}_{\text{rig}}^\dagger} \mathbf{M}_{b,\mu}(V)^{\vee \otimes m_i}$  is isomorphic to a direct sum of  $\mathbf{M}(0,1)$ . By Proposition 3.12 also its direct summand  $\mathbf{M}_{b,\mu}(\tilde{V})$  is isomorphic to  $\mathbf{M}(0,1)^{\oplus \dim \tilde{V}}$  as desired.  $\square$

**Proposition 5.3.** *The set  $\check{\mathcal{F}}_b^0$  is invariant under the group  $J(\mathbb{Q}_p)$  from (1.3) and contained in  $\check{\mathcal{F}}_b^{\text{wa}}$  with  $\check{\mathcal{F}}_b^0(L) = \check{\mathcal{F}}_b^{\text{wa}}(L)$  for any finite extension  $L/\check{E}$ .*

*Proof.* Let  $\mu \in \check{\mathcal{F}}_b^0$  be an analytic point and set  $L = \mathcal{H}(\mu)$ . Consider a faithful representation  $V$  as in (5.1) and let  $\underline{D}_\mu = \underline{D}_{b,\mu}(V) = (D, \varphi_D, Fil_\mu^\bullet D_L)$ . Let  $D' \subset D$  be a  $\varphi_D$ -stable  $K_0$ -subspace and let  $Fil_\mu^i D'_L := D'_L \cap Fil_\mu^i D_L$ . We have to show that  $t_H(\underline{D}') \leq t_N(\underline{D}')$  for the subobject  $\underline{D}' = (D', \varphi_D|_{D'}, Fil_\mu^\bullet D'_L) \subset \underline{D}_\mu$  with equality if  $\underline{D}' = \underline{D}_\mu$ . Consider the  $\varphi$ -submodule  $\mathbf{M}' := \mathbf{M}(\underline{D}') \subset \mathbf{M}(\underline{D}_\mu) = \mathbf{M}_\mu$ . Then

$$\text{rk } \mathbf{M}' \cdot \text{wt } \mathbf{M}' = \deg \mathbf{M}' = t_N(\underline{D}') - t_H(\underline{D}')$$

by Theorem 4.5. Since  $\mu \in \check{\mathcal{F}}_b^0$  we have  $\mathbf{M}_\mu \cong \mathbf{M}(0,1)^{\oplus \dim D}$  and hence  $t_H(\underline{D}_\mu) = t_N(\underline{D}_\mu)$ . Moreover,  $\text{wt } \mathbf{M}' \geq \text{wt } \mathbf{M}_\mu = 0$  by Proposition 3.5 proving  $t_N(\underline{D}') \geq t_H(\underline{D}')$ . This shows that  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}_b^{\text{wa}}$ . The equality on  $L$ -valued points follows from Theorem 4.9.  $\square$

We expect that the equality on  $L$ -valued points holds for many more  $L$ .

**Conjecture 5.4.** *If  $L/\check{E}$  is a complete valued extension such that the value group of  $L$  is finitely generated then  $\check{\mathcal{F}}_b^0(L) = \check{\mathcal{F}}_b^{wa}(L)$ .*

We will see in a moment that  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}_b^{wa}$  is open. We also make the

**Conjecture 5.5.**  *$\check{\mathcal{F}}_b^0$  is arcwise connected.*

We give evidence for these two conjectures in Section 9. In particular we show in Proposition 9.1 that Conjecture 5.5 follows from Conjecture 5.4.

**Theorem 5.6.** *The set  $\check{\mathcal{F}}_b^0$  is a paracompact open  $\check{E}$ -analytic subspace of  $\check{\mathcal{F}}^{\text{an}}$ . If  $b$  is decent with the integer  $s$ ; cf. (1.6); then  $\check{\mathcal{F}}_b^0$  has a natural structure of paracompact open  $E_s$ -analytic subspace of  $(\mathcal{F} \otimes_E E_s)^{\text{an}}$  from which it arises by base change to  $\check{E}$ .*

*Proof.* (a) Since  $\check{\mathcal{F}}^{\text{an}}$  has a finite covering by affine spaces, all its open  $\check{E}$ -analytic subspaces are paracompact by Lemma A.3. To prove that  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}^{\text{an}}$  is open consider a faithful representation  $V$  of dimension  $h$  as in (5.1) and let  $G' = \text{GL}(V)$ . This defines a closed embedding  $\mathcal{F} \hookrightarrow \mathcal{F}' := \text{Flag}(V) \otimes_{\mathbb{Q}_p} E$  into the flag variety from (1.2). Here  $\text{Flag}(V)$  is a Grassmannian isomorphic to  $G'/S'$  where  $S' = \text{Stab}_{G'}(V_0)$  is the stabilizer of an appropriate subspace  $V_0$  of  $V$ . By definition  $\check{\mathcal{F}}_b^0 = \check{\mathcal{F}}^{\text{an}} \cap \check{\mathcal{F}}_b'^0$ . So it suffices to prove the theorem for  $G'$  instead of  $G$ . Since  $G'$  is connected we may assume by [Kot85] that  $b$  is decent, say with integer  $s$ . We let  $\mathcal{F}'_s{}^{\text{an}} := (\mathcal{F}' \otimes_E E_s)^{\text{an}}$  and define the subset  $\check{\mathcal{F}}_b'^0 \subset \mathcal{F}'_s{}^{\text{an}}$  by the same condition as in (5.2). We show that it is open. For  $\mu \in \mathcal{F}'_s{}^{\text{an}}$  we set  $\underline{D}_\mu = \underline{D}_{b,\mu}(V)$ . Our assumption (5.1) implies  $t_N(\underline{D}_\mu) = t_H(\underline{D}_\mu)$  by Lemma 5.1. Choose an integer  $e \geq (\dim V) - 1$  which is a multiple of  $s$ . Since  $\underline{D}_\mu$  has Hodge-Tate weights  $-n$  and  $-n+1$ , the set  $\mathcal{F}'_s{}^{\text{an}} \setminus \check{\mathcal{F}}_b'^0$  is by Proposition 4.10 equal to

$$\left\{ \begin{array}{l} \mu \in \mathcal{F}'_s{}^{\text{an}} \text{ analytic points : there exists an algebraically closed complete} \\ \text{extension } C \text{ of } \mathcal{H}(\mu) \text{ and an element } x \in \mathbf{H}_{\varphi^e}^0(\mathbf{M}(1,1) \otimes [e]_*(t^{-n}\mathbf{D})), x \neq 0 \\ \text{with } \theta_{t^{-n}\mathbf{D}}(\varphi_{t^{-n}\mathbf{D}}^m(x)) \in (\text{Fil}_\mu^m V_{\mathcal{H}(\mu)}) \otimes_{\mathcal{H}(\mu)} C \text{ for all } m = 0, \dots, e-1 \end{array} \right\}.$$

Here  $\mathbf{D} := \mathbf{D}(\underline{D}_\mu) = (D, \varphi_D) \otimes_{K_0} \tilde{\mathbf{B}}_{\text{rig}}^\dagger(C)$  is the  $\varphi$ -module over  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger(C)$  associated with  $\underline{D}_\mu$  in Definition 4.2 and  $[e]_*(t^{-1}\mathbf{D}) = (D, p^{-e}\varphi_D^e) \otimes_{K_0} \tilde{\mathbf{B}}_{\text{rig}}^\dagger(C)$ .

(b) We identify  $V \otimes_{\mathbb{Q}_p} \mathbb{A}_{E_s}^1$  with affine  $h$ -space  $\mathbb{A}_{E_s}^h$  over  $E_s$ . For  $\zeta \in E_s$  consider the  $E_s$ -analytic polydisc  $\mathbb{D}(\zeta)^{eh} = \mathcal{M}(E_s \langle \frac{w_{im}}{\zeta} : i = 1, \dots, h; m = 0, \dots, e-1 \rangle) \subset (\mathbb{A}_{E_s}^h)^e$  with radii  $(|\zeta|, \dots, |\zeta|)$ . We will construct in (c) below a constant  $\zeta \in E_s$  and a compact subset  $Z$  of  $\mathbb{D}(\zeta)^{eh}$  with the following property: If  $C$  is an algebraically closed complete extension of  $E_s$  and  $x \in \mathbf{H}_{\varphi^e}^0(\mathbf{M}(1,1) \otimes [e]_*(t^{-n}\mathbf{D}))$  with  $x \neq 0$  then for some integer  $N$

$$(w_m)_{m=0}^{e-1} := \left( (w_{1m}, \dots, w_{hm})^T \right)_{m=0}^{e-1} := \left( p^N \theta_{t^{-n}\mathbf{D}}(\varphi_{t^{-n}\mathbf{D}}^m(x)) \right)_{m=0}^{e-1}$$

is a  $C$  valued point of  $Z$  and  $Z$  consists precisely of those points.

Now let  $G'_s{}^{\text{an}}$  be the  $E_s$ -analytic space associated with the group scheme  $G' \otimes_{\mathbb{Q}_p} E_s$  and consider the morphism of  $E_s$ -analytic spaces

$$\begin{aligned} \beta : G'_s{}^{\text{an}} \times_{E_s} \mathbb{D}(\zeta)^{eh} &\longrightarrow (\mathbb{A}_{E_s}^h)^e \cong (V \otimes_{\mathbb{Q}_p} \mathbb{A}_{E_s}^1)^e \\ (g, (w_m)_{m=0}^{e-1}) &\longmapsto (g^{-1}w_m)_{m=0}^{e-1}. \end{aligned}$$

Let  $Y$  be the closed subset of  $G'_s{}^{\text{an}} \times_{E_s} \mathbb{D}(\zeta)^{eh}$  defined by the condition that  $(w_m)_{m=0}^{e-1}$  belongs to  $Z$  and that  $\beta(Y) \subset (V_0 \otimes_{\mathbb{Q}_p} \mathbb{A}_{E_s}^1)^e$ . Furthermore consider the projection map

$$pr_1 : G'_s{}^{\text{an}} \times_{E_s} \mathbb{D}(\zeta)^{eh} \longrightarrow G'_s{}^{\text{an}}$$

onto the first factor and the canonical map  $\gamma : G'_s{}^{\text{an}} \rightarrow \mathcal{F}'_s{}^{\text{an}}$  coming from the isomorphism  $\mathcal{F}'_s \cong G'_s / \text{Stab}_{G'_s}(V_0)$ . Then  $\mu \in \mathcal{F}'_s{}^{\text{an}}$  does not belong to  $\mathcal{F}'_b{}^0$  if and only if  $\mu \in \gamma \circ pr_1(Y)$ . Since  $\mathbb{D}(\zeta)^{eh}$  is quasi-compact the projection  $pr_1$  is a proper map of topological Hausdorff spaces by [Beh90, Proposition 3.3.2]. Thus in particular it is closed and  $pr_1(Y)$  is closed. Note that  $\mathcal{F}'_s{}^{\text{an}}$  carries the quotient topology under  $\gamma$  since  $\gamma$  is a smooth morphism of schemes, hence open by [Beh93, Proposition 3.5.8 and Corollary 3.7.4]. Since by construction  $pr_1(Y) = \gamma^{-1}(\gamma \circ pr_1(Y))$  we conclude that  $\mathcal{F}'_b{}^0 = \mathcal{F}'_s{}^{\text{an}} \setminus \gamma \circ pr_1(Y)$  is open in  $\mathcal{F}'_s{}^{\text{an}}$  as desired.

(c) It remains to construct the compact set  $Z$ . Since  $b$  is decent, the  $\varphi^e$ -module  $\mathbf{M}(1, 1) \otimes [e]_*(t^{-n}\mathbf{D})$  is isomorphic to  $\bigoplus_{i=1}^h \mathbf{M}(-c_i, 1)$  for suitable integers  $c_i$ . We assume that the identification of  $V \otimes_{\mathbb{Q}_p} \mathbb{A}_{E_s}^1$  with  $\mathbb{A}_{E_s}^h$  in (b) was chosen compatible with this direct sum decomposition. Let  $c_1, \dots, c_k > 0 = c_{k+1} = \dots = c_\ell > c_{\ell+1}, \dots, c_h$ . Since all Newton slopes of  $\mathbf{D}$ , respectively  $t^{-n}\mathbf{D}$ , lie in  $[n-1, n]$ , respectively  $[-1, 0]$ , we have  $e-1 \geq c_i \geq -1$  for all  $i$ . Then by Propositions 3.8 and 3.12

$$\begin{aligned} \mathrm{H}_{\varphi^e}^0(\mathbf{M}(1, 1) \otimes [e]_*(t^{-n}\mathbf{D})) &\cong \bigoplus_{i=1}^k \left\{ \sum_{\nu \in \mathbb{Z}} p^{c_i \nu} \sum_{j=0}^{c_i-1} p^j \varphi^{-e\nu}([u_{ij}]) : u_{ij} \in \tilde{\mathbf{E}}, v_{\mathbf{E}}(u_{ij}) > 0 \right\} \\ &\oplus \bigoplus_{i=k+1}^{\ell} W(\mathbb{F}_{p^e})\left[\frac{1}{p}\right] \\ &\oplus \bigoplus_{i=\ell+1}^h (0) \quad . \end{aligned}$$

For  $1 \leq i \leq k, 0 \leq j \leq c_i - 1$  and all integers  $\lambda \geq 0$  consider the compact  $E_s$ -analytic spaces which we just view as compact sets

$$\begin{aligned} U_{ij}^{(0)} &:= \mathcal{M}(E_s \langle \frac{u_{ij}^{(0)}}{p} \rangle) = \{ |u_{ij}^{(0)}| \leq |p| \} \quad \text{and} \\ U_{ij}^{(\lambda)} &:= \mathcal{M}(E_s \langle u_{ij}^{(\lambda)} \rangle) = \{ |u_{ij}^{(\lambda)}| \leq 1 \} \quad \text{for } \lambda > 0. \end{aligned}$$

Then the sets

$$\begin{aligned} U_{ij} &:= \left\{ (u_{ij}^{(\lambda)})_{\lambda \in \mathbb{N}_0} \in \prod_{\lambda \in \mathbb{N}_0} U_{ij}^{(\lambda)} : (u_{ij}^{(\lambda+1)})^p = u_{ij}^{(\lambda)} \text{ for all } \lambda \geq 0 \right\} \quad \text{and} \\ U &:= \prod_{i=1}^k \prod_{j=0}^{c_i-1} U_{ij} \times \prod_{i=k+1}^{\ell} W(\mathbb{F}_{p^e}) \times \prod_{i=\ell+1}^h \{0\} \end{aligned}$$

are compact by Tychonoff's theorem. For an arbitrary algebraically closed complete extension  $C$  of  $E_s$  consider a  $C$ -valued point  $u$  of  $U$  given by

$$\left( (u_{ij}^{(\lambda)})_{\lambda \in \mathbb{N}_0} \right)_{i=1, \dots, k; j=0, \dots, c_i-1}, (a_i)_{i=k+1, \dots, \ell}, (0)_{i=\ell+1, \dots, h}$$

with  $u_{ij}^{(\lambda)} \in C$  and  $a_i \in W(\mathbb{F}_{p^e})$ . We assign to  $u$  the  $C$ -valued point  $y$  of  $\mathbb{A}_{E_s}^{eh}$  with  $(w_m)_{m=0}^{e-1} = (\theta_{t^{-n}\mathbf{D}}(\varphi_{t^{-n}\mathbf{D}}^m(x)))_{m=0}^{e-1}$  where  $x$  is the element of  $H_{\varphi^e}^0(\mathbf{M}(1, 1) \otimes [e]_*(t^{-n}\mathbf{D}))$  associated with the  $u_{ij}^{(\lambda)}$  and  $a_i$ . In concrete terms this means

$$w_m := b \cdot \varphi(b) \cdots \varphi^{m-1}(b) \cdot \left( \begin{array}{c} \left( \sum_{j=0}^{c_i-1} p^j \left( \sum_{\nu>0} p^{c_i\nu} u_{ij}^{(e\nu-m)} + \sum_{\nu \leq 0} p^{c_i\nu} (u_{ij}^{(0)})^{p^{m-\nu e}} \right) \right)_{i=1}^k \\ (\varphi^m(a_i))_{i=k+1}^\ell \\ (0)_{i=\ell+1}^h \end{array} \right)$$

for  $m = 0, \dots, e-1$ . This defines a map  $\alpha : U \rightarrow \mathbb{A}_{E_s}^{eh}$ ,  $u \mapsto y$  of topological Hausdorff spaces. We claim that  $\alpha$  is continuous. By definition of the topology on  $\mathbb{A}_{E_s}^{eh}$  (see Appendix A) the map  $\alpha$  is continuous if and only if for any polynomial  $f \in E_s[w_{im} : 1 \leq i \leq h, 0 \leq m \leq e-1]$  and any open interval  $I \subset \mathbb{R}$  the preimage under  $\alpha$  of the open set

$$W = \{ y \in \mathbb{A}_{E_s}^{eh} : |f|_y \in I \}$$

is open in  $U$ . Since  $|u_{ij}^{(0)}|_u \leq |p|$  there is a constant  $\zeta \in E_s$  such that  $|w_{im}|_{\alpha(u)} \leq |\zeta|$  for all  $i$  and  $m$  and all  $u \in U$ . Write  $f = \sum_{\underline{n}} b_{\underline{n}} \underline{w}^{\underline{n}}$  where  $b_{\underline{n}} \in E_s$  for every multi index  $\underline{n} \in \mathbb{N}_0^{eh}$ . If we set  $w_{im} = w'_{im} + w''_{im}$  we obtain from the Taylor expansion of  $f$  in powers of  $\underline{w}''$  a bound  $\delta$  such that for all  $y \in \mathbb{A}_{E_s}^{eh}$  the condition  $|w''_{im}|_y \leq \delta$  for all  $i, m$  implies

$$|f(\underline{w})|_y \in I \text{ and } |\underline{w}|_y \leq |\zeta| \iff |f(\underline{w}')|_y \in I \text{ and } |\underline{w}'|_y \leq |\zeta|.$$

Now we fix a positive integer  $r$  and set for all  $m = 0, \dots, e-1$

$$(w''_{im})_{i=1}^h := b \cdot \varphi(b) \cdots \varphi^{m-1}(b) \cdot \left( \begin{array}{c} \left( \sum_{j=0}^{c_i-1} p^j \sum_{\nu>r} p^{c_i\nu} u_{ij}^{(e\nu-m)} \right)_{i=1}^k \\ (0)_{i=k+1}^\ell \\ (0)_{i=\ell+1}^h \end{array} \right) \quad \text{and}$$

$$(w'_{im})_{i=1}^h := b \cdot \varphi(b) \cdots \varphi^{m-1}(b) \cdot \left( \begin{array}{c} \left( \sum_{j=0}^{c_i-1} p^j \sum_{\nu \leq r} p^{c_i\nu} (u_{ij}^{(er)})^{p^{m+(r-\nu)e}} \right)_{i=1}^k \\ (\varphi^m(a_i))_{i=k+1}^\ell \\ (0)_{i=\ell+1}^h \end{array} \right).$$

Since  $|u_{ij}^{(\lambda)}|_u < 1$  for any  $\lambda, i, j$  and for any point  $u \in U$  we can find a large enough integer  $r$  such that  $|w''_{im}|_{\alpha(u)} \leq \delta$  for all  $i, m$  and for all  $u \in U$ . This shows that

$$\alpha^{-1}(W) = W' := \{ u \in U : |f(w'_{im})|_{\alpha(u)} \in I \}.$$

To prove that  $W'$  is open in  $U$  observe that the map

$$\begin{aligned} \alpha'_m : U' &:= \prod_{i=1}^k \prod_{j=0}^{c_i-1} U_{ij}^{(er)} \times \prod_{i=k+1}^\ell \mathbb{A}_{E_s}^1 \longrightarrow \mathbb{A}_{E_s}^h \\ & \left( (u_{ij}^{(er)})_{i,j}, (\varphi^m(a_i))_i \right) \longmapsto (w'_{im})_{i=1}^h \end{aligned}$$

is a morphism of  $E_s$ -analytic spaces, and in particular continuous. Furthermore, the projection maps  $\alpha_{ij} : U_{ij} \rightarrow U_{ij}^{(er)}$  and the inclusion

$$\alpha_i : W(\mathbb{F}_{p^s}) \longrightarrow \mathbb{A}_{E_s}^e = \mathcal{M}(E_s[T_0, \dots, T_{e-1}]), \quad a \longmapsto (T_m \mapsto \varphi^m(a))$$

for  $i = k+1, \dots, \ell$  are continuous. Since

$$\alpha' := (\alpha'_0 \times \dots \times \alpha'_{e-1}) \circ \left( \prod_{i=1}^k \prod_{j=0}^{c_i-1} \alpha_{ij} \times \prod_{i=k+1}^{\ell} \alpha_i \right) : U \longrightarrow \mathbb{A}_{E_s}^{eh}$$

maps  $u$  to  $\alpha'(u) = (w'_m)_{m=0}^{e-1}$  the set  $W' = (\alpha')^{-1}(W)$  is open in  $U$  and this proves that  $\alpha$  is continuous.

Now multiplying  $(w_{im})_{i,m}$  with  $p$  amounts to replacing  $u_{ij}^{(\lambda)}$  by  $u_{i,j-1}^{(\lambda)}$  for  $j = 1, \dots, c_i-1$ , and  $u_{i,0}^{(\lambda)}$  by  $(u_{i,c_i-1}^{(\lambda)})^{p^e}$ , and  $a_i$  by  $pa_i$ . Thus we may take

$$Z := \alpha \left( U \setminus \left\{ u \in U : |u_{ij}^{(0)}|_u < |p|^{p^e}, a_i \in pW(\mathbb{F}_{p^e}) \text{ for all } i \text{ and } j \right\} \right).$$

Then  $Z$  is the continuous image of a compact set and satisfies the property required in (b). This proves the theorem.  $\square$

We give an example showing that the inclusion  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}_b^{wa}$  may be strict. Similar examples were independently obtained by A. Genestier and V. Lafforgue. In Section 8 we will determine all  $b \in \mathrm{GL}_n(K_0)$  for which  $\check{\mathcal{F}}_b^0 = \check{\mathcal{F}}_b^{wa}$ .

**Example 5.7.** Let  $G = \mathrm{GL}_5$  and consider the conjugacy class  $\{\mu\}$  of cocharacters containing  $\mu : \mathbb{G}_m \rightarrow G, z \mapsto \mathrm{diag}(z, z, 1, 1, 1)$ . We have  $E = \mathbb{Q}_p$  and  $\mathcal{F} \cong \mathrm{Grass}(2, 5)$  the Grassmannian. Let

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 & p^{-2} \\ p & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & p & 0 \end{pmatrix} \in G(K_0).$$

The element  $b$  is decent with integer  $s = 5$  and  $(G, b, \{\mu\})$  satisfies (5.1) for  $n = 1$ . Since the isocrystal  $(K_0^{\oplus 5}, b \cdot \varphi)$  is simple every cocharacter in  $\{\mu\}$  is weakly admissible, that is  $\check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}_b^{\mathrm{an}}$ .

Let  $\mu \in \check{\mathcal{F}}_b^{\mathrm{an}}$  and let  $C$  be the completion of an algebraic closure of  $\mathcal{H}(\mu)$ . The  $\varphi$ -module  $\mathbf{M}_\mu = \mathbf{M}(K_0^{\oplus 5}, b \cdot \varphi, \mathrm{Fil}_\mu^\bullet \mathcal{H}(\mu)^{\oplus 5})$  constructed from  $b$  and  $\mu$  satisfies  $\mathrm{deg} \mathbf{M}_\mu = 0$  and

$$t^{-1}\mathbf{D} = \mathbf{D} \otimes \mathbf{M}(-1, 1) = \mathbf{M}(-3, 5) \supset \mathbf{M}_\mu \supset \mathbf{D} \cong \mathbf{M}(2, 5)$$

where  $\mathbf{D} = \mathbf{D}(K_0^{\oplus 5}, b \cdot \varphi) = (K_0^{\oplus 5}, b \cdot \varphi) \otimes_{K_0} \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C)$ . Since by Proposition 3.12 the weight of every summand of  $\mathbf{M}_\mu$  lies between  $-3/5$  and  $2/5$ , either  $\mathbf{M}_\mu \cong \mathbf{M}(0, 1)^{\oplus 5}$  or  $\mathbf{M}_\mu \cong \mathbf{M}(-1, 2) \oplus \mathbf{M}(1, 3)$ . (The first entries must sum to  $0 = \mathrm{deg} \mathbf{M}_\mu$  and the second entries must sum to  $5 = \mathrm{rk} \mathbf{M}_\mu$ .) The bad situation  $\mathbf{M}_\mu \cong \mathbf{M}(-1, 2) \oplus \mathbf{M}(1, 3)$  occurs if and only if

$$\begin{aligned} (0) &\neq \mathrm{Hom}_\varphi(\mathbf{M}(-1, 2), \mathbf{M}_\mu) = \\ &= \left\{ f \in \mathrm{Hom}_\varphi(\mathbf{M}(-1, 2), t^{-1}\mathbf{D}) : \theta_{t^{-1}\mathbf{D}}(\mathrm{im} f) \subset \mathrm{Fil}_\mu^1 \mathcal{H}(\mu)^{\oplus 5} \right\}. \end{aligned}$$

One easily checks that  $\mathrm{Hom}_\varphi(\mathbf{M}(-1, 2), \mathbf{M}(-3, 5)) =$

$$= \left\{ A = \begin{pmatrix} \varphi^5(x) & x \\ \varphi^{11}(x) & \varphi^6(x) \\ \varphi^{17}(x) & \varphi^{12}(x) \\ \varphi^{23}(x) & \varphi^{18}(x) \\ \varphi^{29}(x) & \varphi^{24}(x) \end{pmatrix} : x = \sum_{\nu \in \mathbb{Z}} p^\nu \varphi^{-10\nu}([u]) \in H_{\varphi^{10}}^0(\mathbf{M}(1, 1)), \right. \\ \left. u \in \tilde{\mathbf{E}}(C), 0 < v_{\mathbf{E}}(u) < \infty \right\}.$$

Thus the points  $\mu \in \check{\mathcal{F}}^{\mathrm{an}}$  for which the columns of  $\theta_{t-1\mathbf{D}}(A)$  are contained in  $\mathrm{Fil}_\mu^1 \mathcal{H}(\mu)^{\oplus 5}$  do not belong to  $\check{\mathcal{F}}_b^0$  and such points exist! This proves that the inclusion  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}_b^{wa}$  is strict.

Note that  $\mathrm{rk} \theta_{t-1\mathbf{D}}(A) = 2$  and  $\mu$  is uniquely determined by  $A$  and hence by  $u$ . Indeed, assume the contrary and consider  $\wedge^2 A \in \mathrm{Hom}_\varphi(\wedge^2 \mathbf{M}(-1, 2), \wedge^2 \mathbf{M}(-3, 5))$  which is non-zero because  $A$  is an inclusion  $\mathbf{M}(-1, 2) \hookrightarrow \mathbf{M}(-3, 5)$ . Here  $\wedge^2 \mathbf{M}(-1, 2) = \mathbf{M}(-1, 1)$  and  $\wedge^2 \mathbf{M}(-3, 5) \cong \mathbf{M}(-6, 5)^{\oplus 2}$  by [Ked05, Proposition 4.6.3]. Then  $\theta_{\wedge^2 t-1\mathbf{D}}(\wedge^2 A) = \wedge^2 \theta_{t-1\mathbf{D}}(A) = 0$  implies that  $\wedge^2 A = T \circ B$  for  $0 \neq B \in \mathrm{Hom}_\varphi(\mathbf{M}(-1, 1), \mathbf{M}(-1, 5)^{\oplus 2})$  by Corollary 3.11. But since  $-1 < -\frac{1}{5}$ , the later space is zero by Proposition 3.12 which gives the contradiction to our assumption.

In particular the map  $\alpha : \{u \in \tilde{\mathbf{E}} : 0 < |u| < 1\} \rightarrow \check{\mathcal{F}}^{\mathrm{an}}, u \mapsto \mu$  parametrizes the complement  $\check{\mathcal{F}}^{\mathrm{an}} \setminus \check{\mathcal{F}}_b^0$ . The nature of the map  $\alpha$  is somewhat mysterious. It is continuous by similar arguments as in part (c) of the proof of Theorem 5.6 but its image does not meet the dense subset of points  $x \in \check{\mathcal{F}}^{\mathrm{an}}$  with  $\mathcal{H}(x)$  finite over  $K_0$ .

We conclude this section by stating the following theorem which we will prove at the end of Section 6.

**Theorem 5.8.** *For any representation  $V$  of  $G$  as in (5.1) there is a local system  $\mathcal{V}$  of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$  such that the fiber of  $\mathcal{V}$  at any point  $\mu \in \check{\mathcal{F}}_b^0(L)$  for a finite extension  $L/\check{E}$  induces via (1.7) the crystalline Galois representation  $V_{\mathrm{cris}}(\underline{D}_{b,\mu}(V))$  associated with the (weakly) admissible filtered isocrystal  $\underline{D}_{b,\mu}(V) = (V_{K_0}, \rho(b) \cdot \varphi, \mathrm{Fil}_\mu^\bullet V_L)$ .*

**Remark 5.9.** Note that this theorem is weaker than Conjecture 1.3 in two respects. Firstly we do not know in general whether  $V \mapsto \mathcal{V}$  is a tensor functor  $\underline{\mathrm{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-}\underline{\mathrm{Loc}}_{\check{\mathcal{F}}_b^0}$  as required in Conjecture 1.3. However this is true if  $G = \mathrm{GL}_n$  (see Theorem 6.3) or if  $G$  is the group associated to an EL or PEL situation (see Theorem 7.4).

Secondly we do neither know whether  $\check{\mathcal{F}}_b^0$  is connected nor whether it is the largest open subspace of  $\check{\mathcal{F}}_b^{wa}$  with this property. However, we conjecture that this is true.

**Conjecture 5.10.**  *$\check{\mathcal{F}}_b^0$  is the largest open  $\check{E}$ -analytic subspace of  $\check{\mathcal{F}}_b^{wa}$  carrying a local system  $\mathcal{V}$  that has the properties stated in the theorem.*

We give some evidence for this in Section 9.

## 6 Relation with Period Morphisms

For  $G = \mathrm{GL}(V)$  Rapoport and Zink also study a period morphism  $\check{\pi} : \check{\mathcal{G}}^{\mathrm{an}} \rightarrow \check{\mathcal{F}}_b^{wa}$ . We keep the notation of Section 5. In particular let  $\check{b} \in \mathrm{GL}(V)(K_0)$ , let  $\{\check{\mu}\}$  be a conjugacy class of cocharacters  $\mathbb{G}_m \rightarrow \mathrm{GL}(V)$ . Then  $\check{E} = K_0$ . We assume that  $V$  satisfies (5.1) for an

integer  $\tilde{n}$ . We set  $b := p^{1-\tilde{n}} \cdot \tilde{b} \in G(K_0)$  and define  $\mu : \mathbb{G}_m \rightarrow G$  by  $\mu(z) = z^{1-\tilde{n}} \cdot \tilde{\mu}(z)$ . Then  $\underline{D}_{b,\mu}(V) = \underline{D}_{\tilde{b},\tilde{\mu}}(V) \otimes \mathbb{1}(1-\tilde{n})$  and  $\mathbf{M}_{b,\mu}(V) = \mathbf{M}_{\tilde{b},\tilde{\mu}}(V)$  by Lemma 4.3. This implies that the assignment  $\tilde{\mu} \mapsto \mu$  is an isomorphism  $\check{\mathcal{F}}_{\tilde{b}}^0 \xrightarrow{\sim} \check{\mathcal{F}}_b^0$ .

Now  $(G, b, \{\mu\})$  and  $V$  satisfy (5.1) for  $n = 1$ . Therefore  $(V_{K_0}, b \cdot \varphi)$  is the covariant Dieudonné-module  $(D, \varphi_D)$  of a  $p$ -divisible group  $\mathbf{X} = \mathbf{X}_b(V)$  over  $\mathbb{F}_p^{\text{alg}}$ ; see Messing [Mes72]. This means  $D := \mathbb{D}(\mathbf{X})$  is the tensor product with  $K_0$  of the Lie algebra of the universal vector extension of some (any) lift of  $\mathbf{X}$  to a  $p$ -divisible group over  $W(\mathbb{F}_p^{\text{alg}})$ . On  $\mathbf{X}$  there is the Verschiebung morphism  $Ver_{\mathbf{X}} : \text{Frob}_p^* \mathbf{X} \rightarrow \mathbf{X}$ . By the crystalline nature of  $\mathbb{D}(\mathbf{X})$  it induces the  $\varphi$ -semilinear isomorphism  $\varphi_D := \mathbb{D}(Ver_{\mathbf{X}}) \circ \varphi : D \rightarrow D$ . Moreover, (5.1) and Lemma 5.1 imply  $\dim \mathbf{X} = \dim_{K_0} D - t_N(\underline{D}_{b,\mu}(V)) = \dim_{K_0} D - t_H(\underline{D}_{b,\mu}(V)) = \dim V - \dim_L \text{Fil}^1 D_L =: d$  and  $\check{\mathcal{F}}^{\text{an}}$  is the Grassmannian of  $(\dim V - d)$ -dimensional subspaces of  $V$ .

The period morphism is constructed as follows. Let  $\mathcal{N}ilp_{W(\mathbb{F}_p^{\text{alg}})}$  be the category of  $W(\mathbb{F}_p^{\text{alg}})$ -schemes on which  $p$  is locally nilpotent. For  $S \in \mathcal{N}ilp_{W(\mathbb{F}_p^{\text{alg}})}$  denote by  $\bar{S}$  the closed subscheme defined by the ideal  $p\mathcal{O}_S$ . We set  $\mathbf{X}_{\bar{S}} := \mathbf{X} \times_{\mathbb{F}_p^{\text{alg}}} \bar{S}$ .

**Proposition 6.1** ([RZ96, Theorem 2.16]). *The contravariant functor  $\mathcal{N}ilp_{W(\mathbb{F}_p^{\text{alg}})} \rightarrow \text{Sets}$*

$$S \longmapsto \left\{ \begin{array}{l} \text{Isomorphism classes of } (X, \eta : \mathbf{X}_{\bar{S}} \rightarrow X_{\bar{S}}) \text{ where} \\ X \text{ is a } p\text{-divisible group over } S \text{ and} \\ \eta \text{ is a quasi-isogeny over } \bar{S} \text{ to } X_{\bar{S}} = X \times_S \bar{S} \end{array} \right\}$$

is representable by a formal scheme  $\check{\mathcal{G}}$  locally formally of finite type over  $\text{Spf } W(\mathbb{F}_p^{\text{alg}})$ .

Rapoport and Zink also study formal moduli schemes corresponding to additional data on  $(X, \eta)$  of type EL and PEL; see Section 7. Their period morphisms are derived from the following prototype.

Let  $(X, \eta : \mathbf{X}_{\check{\mathcal{G}}} \rightarrow X_{\check{\mathcal{G}}})$  be the universal  $p$ -divisible group with quasi-isogeny over  $\check{\mathcal{G}}$ . Let  $\mathbb{D}(X)_{\check{\mathcal{G}}}$  be the Lie algebra of the universal vector extension of  $X$  over  $\check{\mathcal{G}}$ . It sits in the exact sequence

$$0 \longrightarrow (\text{Lie } X^\vee)^\vee \longrightarrow \mathbb{D}(X)_{\check{\mathcal{G}}} \longrightarrow \text{Lie } X \longrightarrow 0.$$

Let  $\check{\mathcal{G}}^{\text{an}}$  be the  $K_0$ -analytic space associated with  $\check{\mathcal{G}}$ ; see Theorem A.2. The quasi-isogeny  $\eta$  induces an isomorphism

$$\mathbb{D}(\eta) : D \otimes_{K_0} \mathcal{O}_{\check{\mathcal{G}}^{\text{an}}} = \mathbb{D}(\mathbf{X})_{K_0} \otimes_{K_0} \mathcal{O}_{\check{\mathcal{G}}^{\text{an}}} \xrightarrow{\sim} \mathbb{D}(X)_{\check{\mathcal{G}}^{\text{an}}}$$

see [RZ96, Proposition 5.15]. The kernel of the morphism

$$V_{K_0} \otimes_{K_0} \mathcal{O}_{\check{\mathcal{G}}^{\text{an}}} \xrightarrow{\sim} \mathbb{D}(X)_{\check{\mathcal{G}}^{\text{an}}} \longrightarrow (\text{Lie } X)_{\check{\mathcal{G}}^{\text{an}}}$$

defines the  $\check{\mathcal{G}}^{\text{an}}$ -valued point  $\mathbb{D}(\eta)^{-1}(\text{Lie } X^\vee)_{\check{\mathcal{G}}^{\text{an}}}^\vee$  of the Grassmannian  $\check{\mathcal{F}}^{\text{an}}$ . This is the desired *period morphism*  $\check{\pi} : \check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{F}}^{\text{an}}$ . It factors through  $\check{\mathcal{F}}_b^{wa}$ .

**Proposition 6.2.** *The period morphism factors through  $\check{\mathcal{F}}_b^0$  and is surjective on the level of rigid analytic points.*

*Proof.* We fix the tautological representation  $V$  of  $G$ , set  $D = V_{K_0}$ , and consider  $\check{\mathcal{F}}^{\text{an}}$  as the Grassmannian of  $(\dim V - d)$ -dimensional subspaces of  $V$ . Let  $x \in \check{\mathcal{G}}^{\text{an}}$  be an analytic point and let  $\mu = \check{\pi}(x) \in \check{\mathcal{F}}^{\text{an}}$ . Let  $L = \mathcal{H}(x)$  and let  $C$  be the completion of an algebraic closure of  $L$ . Let  $X_x$  be the fiber of the universal  $p$ -divisible group  $X$  at  $x$  and consider the Tate module  $T_p X_x$  of  $X_x$ . An element  $\lambda \in T_p X_x$  corresponds to a morphism of  $p$ -divisible groups  $\lambda : \mathbb{Q}_p/\mathbb{Z}_p \rightarrow X_{\mathcal{O}_C}$  over  $\mathcal{O}_C$ . By functoriality of the universal vector extension this yields the following diagram of  $C$ -vector spaces

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_C & \xlongequal{\quad} & \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \mathbb{D}(\lambda)_C & & \\ 0 & \longrightarrow & (\text{Fil}_\mu^1 D_L) \otimes_L C & \longrightarrow & \mathbb{D}(X)_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} C & \longrightarrow & (\text{Lie } X_x)_C \longrightarrow 0. \end{array} \quad (6.1)$$

Note that  $\text{Lie } \mathbb{Q}_p/\mathbb{Z}_p = (0)$  and that  $\text{Fil}_\mu^1 D_L$  is the  $L$ -subspace of  $D_L$  associated with  $X_x$  via the period morphism. We also obtain a morphism of crystals  $\mathbb{D}(\lambda) : \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \mathbb{D}(X_{\mathcal{O}_C})$  which we evaluate on the pd-thickening  $\mathbf{B}_{\text{cris}}^+(C)$  of  $\mathcal{O}_C$ . Since  $\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbf{B}_{\text{cris}}^+(C)} = \mathbf{B}_{\text{cris}}^+(C)$  (because the universal vector extension of  $\mathbb{Q}_p/\mathbb{Z}_p$  over  $\mathbf{B}_{\text{cris}}^+(C)$  is obtained from the sequence  $0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0$  by pushout via  $\mathbb{Z}_p \rightarrow \mathbf{B}_{\text{cris}}^+(C)$ ) we obtain a morphism

$$\begin{array}{ccc} T_p X_x \otimes_{\mathbb{Z}_p} \mathbf{B}_{\text{cris}}^+(C) & \longrightarrow & \mathbb{D}(X_x)_{\mathbf{B}_{\text{cris}}^+(C)} \xleftarrow{\sim} D \otimes_{K_0} \mathbf{B}_{\text{cris}}^+(C). \\ \lambda \otimes a & \longmapsto & \mathbb{D}(\lambda)(a) \end{array}$$

Here the isomorphism on the right arises from the quasi-isogeny  $\eta_x$  by the same reasoning as above. By Faltings's [Fal99, Theorem 7] the morphism on the left is injective. Since the elements of  $T_p X_x$  are  $\varphi$ -invariant inside  $T_p X_x \otimes_{\mathbb{Z}_p} \mathbf{B}_{\text{cris}}^+(C)$  and  $\tilde{\mathbf{B}}_{\text{rig}}^+(C)$  equals  $\bigcap_{n \in \mathbb{N}_0} \varphi^n \mathbf{B}_{\text{cris}}^+(C)$  we get a monomorphism

$$T_p X_x \otimes_{\mathbb{Z}_p} \tilde{\mathbf{B}}_{\text{rig}}^+(C) \hookrightarrow D \otimes_{K_0} \tilde{\mathbf{B}}_{\text{rig}}^+(C).$$

It gives rise to a monomorphism

$$\begin{array}{ccccc} T_p X_x & \hookrightarrow & T_p X_x \otimes_{\mathbb{Z}_p} \tilde{\mathbf{B}}^{\{0,1\}}(C) & \hookrightarrow & D \otimes_{K_0} \tilde{\mathbf{B}}^{\{0,1\}}(C) \\ \lambda & \longmapsto & \lambda \otimes 1 & \longmapsto & \mathbb{D}(\lambda)(1) \end{array}$$

since  $\tilde{\mathbf{B}}^{\{0,1\}}(C)$  is a flat  $\tilde{\mathbf{B}}_{\text{rig}}^+(C)$ -algebra. Consider the morphism  $\theta : D \otimes_{K_0} \tilde{\mathbf{B}}^{\{0,1\}} \rightarrow D \otimes_{K_0} C$ . Diagram (6.1) shows that  $\theta(T_p X_x) \subset (\text{Fil}_\mu^1 D_L) \otimes_L C$  and so  $T_p X_x \otimes_{\mathbb{Z}_p} \tilde{\mathbf{B}}_{\text{rig}}^+(C)$  is in fact contained in the  $\varphi$ -module  $\mathbf{M}_\mu := \mathbf{M}(D, \varphi_D, \text{Fil}_\mu^1 D_L)$  by Proposition 4.1. But then  $\mathbf{M}_\mu \cong \mathbf{M}(0, 1)^{\oplus \dim V}$ . Namely, assume  $\mathbf{M}_\mu \cong \bigoplus_j \mathbf{M}(c_j, d_j)$  with  $c_1 > 0$  (note that  $\sum_j c_j = \deg \mathbf{M}_\mu = 0$ ). Since the elements of  $T_p X_x$  are  $\varphi$ -invariant and  $H_\varphi^0(\mathbf{M}(c_1, d_1)) = (0)$  by Proposition 3.12 the projection

$$T_p X_x \otimes_{\mathbb{Z}_p} \tilde{\mathbf{B}}_{\text{rig}}^+(C) \longrightarrow \mathbf{M}(c_1, d_1)$$

is zero. Thus  $T_p X_x \otimes_{\mathbb{Z}_p} \tilde{\mathbf{B}}_{\text{rig}}^+(C) \hookrightarrow \bigoplus_{j>1} \mathbf{M}(c_j, d_j)$  but this is impossible since we have  $\text{rk}_{\mathbb{Z}_p} T_p X_x = \dim V > \text{rk } \bigoplus_{j>1} \mathbf{M}(c_j, d_j)$ . This proves that  $\check{\pi}$  factors through  $\check{\mathcal{F}}_b^0$ .

It remains to show that  $\check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{F}}_b^0 \hookrightarrow \check{\mathcal{F}}_b^{wa}$  is surjective on the level of rigid analytic points. This follows from the Colmez-Fontaine Theorem [CF00] and Kisin's proof [Kis06, Theorem 0.3] of Fontaine's conjecture that every crystalline representation with Hodge-Tate weights 0 and  $-1$  arises from a  $p$ -divisible group. See also Breuil [Bre00, Theorem 1.4].  $\square$

Using a result of Faltings [Fal10] we show in Theorem 6.3 below that  $\check{\mathcal{F}}_b^0$  actually equals the image of the period morphism  $\check{\pi}$ . In order to formulate part (c) of the theorem we recall from Remark 1.6 that with each tensor functor  $\underline{\mathcal{V}} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-Loc}_{\check{\mathcal{F}}_b^0}$  one associates a tower of étale covering spaces  $\{\check{\mathcal{E}}_{\tilde{K}}\}_{\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)}$  of  $\check{\mathcal{F}}_b^0$ . In the situation we treat in this section,  $G$  is the algebraic group  $\text{GL}(V)$  over  $\mathbb{Q}_p$  and in the notation introduced before Corollary 1.5 we use Hilbert 90 (see [Mil80, Proposition III.4.9]) to identify  $\tilde{\omega} = \omega_0$ ,  $\tilde{V} = \tilde{\omega}(V) = \omega_0(V) = V$ , and  $\tilde{G} = G = \text{GL}(\tilde{V})$ . The situation will be more general in the next section. Similarly to these covering spaces Rapoport and Zink [RZ96, 5.32–5.39] constructed a tower of coverings  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$  of  $\check{\mathcal{G}}^{\text{an}}$ : For any compact open subgroup  $\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)$  let  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$  be the  $\check{E}$ -analytic space representing  $\tilde{K}$ -level structures on the universal  $p$ -divisible group  $X_{\check{\mathcal{G}}^{\text{an}}}$ , that is classes modulo  $\tilde{K}$  of trivializations  $\alpha : \tilde{V} \xrightarrow{\sim} (V_p X_{\check{\mathcal{G}}^{\text{an}}})_{\tilde{\mu}}$  of the rational Tate module (at some geometric base point  $\tilde{\mu}$  of  $\check{\mathcal{F}}_b^0$ ), such that  $\alpha \bmod \tilde{K}$  is invariant under the étale fundamental group of  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$ . The  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$  do not depend on the choice of  $\tilde{\mu}$ . After fixing a  $\mathbb{Z}_p$ -lattice  $\tilde{\Lambda}$  in  $\tilde{V}$  and a preimage  $\tilde{x} \in \check{\mathcal{G}}^{\text{an}}$  of  $\tilde{\mu}$  we can identify  $\check{\mathcal{G}}_{\text{GL}(\tilde{\Lambda})}^{\text{an}}$  with  $\check{\mathcal{G}}^{\text{an}}$  by considering on  $\check{\mathcal{G}}^{\text{an}}$  the residue class of trivializations  $(\alpha : \tilde{\Lambda} \xrightarrow{\sim} (T_p X_{\check{\mathcal{G}}^{\text{an}}})_{\tilde{x}}) \bmod \text{GL}(\tilde{\Lambda})$ . For  $\tilde{K} \subset \text{GL}(\tilde{\Lambda})$  the space  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$  is then a finite étale covering space of  $\check{\mathcal{G}}^{\text{an}}$ . For any  $\tilde{K}$  and any  $g \in \tilde{G}(\mathbb{Q}_p)$  there are  $\check{\mathcal{F}}_b^0$ -isomorphisms  $i_{\tilde{K}}(g) : \check{\mathcal{G}}_{\tilde{K}}^{\text{an}} \xrightarrow{\sim} \check{\mathcal{G}}_{g^{-1}\tilde{K}g}^{\text{an}}$  given by  $(\alpha \bmod \tilde{K}) \mapsto (\alpha g \bmod g^{-1}\tilde{K}g)$ . This proves that the  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$  are  $\check{E}$ -analytic spaces and Rapoport and Zink [RZ96, 5.39] show that they are independent of the choice of  $\tilde{\Lambda}$  and  $\tilde{x}$ . On the tower  $(\check{\mathcal{G}}_{\tilde{K}}^{\text{an}})_{\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)}$  the group  $\tilde{G}(\mathbb{Q}_p)$  acts through Hecke correspondences; see [RZ96, 5.34].

**Theorem 6.3.** *Let  $G$  be the algebraic group  $\text{GL}(V)$  over  $\mathbb{Q}_p$  and set  $\tilde{G} = G$ .*

- (a) *The image  $\check{\pi}(\check{\mathcal{G}}^{\text{an}})$  of the period morphism  $\check{\pi}$  for  $\text{GL}(V)$  equals  $\check{\mathcal{F}}_b^0$ .*
- (b) *The rational Tate module  $V_p X_{\check{\mathcal{G}}^{\text{an}}}$  of the universal  $p$ -divisible group  $X_{\check{\mathcal{G}}^{\text{an}}}$  over  $\check{\mathcal{G}}^{\text{an}}$  descends to a local system  $\mathcal{V}$  of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$ . This induces a tensor functor  $\underline{\mathcal{V}}$  from  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  to the category  $\mathbb{Q}_p\text{-Loc}_{\check{\mathcal{F}}_b^0}$  of local systems of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$  satisfying (1.8) of Conjecture 1.3 and  $\underline{\mathcal{V}}(V) \cong \mathcal{V}$ .*
- (c) *The tower of  $\check{E}$ -analytic spaces  $(\check{\mathcal{G}}_{\tilde{K}}^{\text{an}})_{\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)}$  is canonically isomorphic over  $\check{\mathcal{F}}_b^0$  in a Hecke equivariant way to the tower of étale covering spaces  $(\check{\mathcal{E}}_{\tilde{K}})_{\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)}$  of  $\check{\mathcal{F}}_b^0$  associated with the tensor functor  $\underline{\mathcal{V}}$ . In particular,  $\check{\mathcal{G}}^{\text{an}}$  is (non-canonically) isomorphic to the space of  $\mathbb{Z}_p$ -lattices inside the local system  $\mathcal{V}$ .*

A previously known example for this theorem is the Lubin-Tate situation where  $\check{\mathcal{F}}_b^0 = \check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}^{\text{an}} = \mathbb{P}_{K_0}^{h-1}$ . In this case the theorem was proved by de Jong [dJ95, Proposition 7.2].

*Proof of Theorem 6.3.* (a) The inclusion  $\check{\pi}(\check{\mathcal{G}}^{\text{an}}) \subset \check{\mathcal{F}}_b^0$  was proved in Proposition 6.2. So let now  $\mu \in \check{\mathcal{F}}_b^0$  and let  $C$  be the completion of an algebraic closure of  $L := \mathcal{H}(\mu)$ . Consider

the morphisms

$$\alpha : (\tilde{\mathbf{B}}_{\text{rig}}^\dagger) \otimes_{\mathbb{Q}_p} V = \mathbf{M}(0, 1) \otimes_{\mathbb{Q}_p} V \cong \mathbf{M}_\mu \hookrightarrow t^{-1}\mathbf{D} \quad \text{and} \quad \beta : \mathbf{D} \hookrightarrow \mathbf{M}_\mu \cong (\tilde{\mathbf{B}}_{\text{rig}}^\dagger) \otimes_{\mathbb{Q}_p} V.$$

Let  $h = \dim V$  and fix a basis of  $V$ . This induces a basis of  $\mathbf{D} = V \otimes_{\mathbb{Q}_p} \tilde{\mathbf{B}}_{\text{rig}}^\dagger$ . With respect to this basis,  $\alpha$  and  $\beta$  are represented by matrices  $A, B \in M_h(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)$  satisfying  $AB = t \text{Id}_h$ . Then [Ber04, Proposition I.4.1] implies that  $A, B \in M_h(\tilde{\mathbf{B}}_{\text{rig}}^\dagger) \subset M_h(\mathbf{B}_{\text{cris}}^+)$ . So  $A$  defines an isomorphism  $\mathbf{B}_{\text{cris}}^{\oplus h} \xrightarrow{\sim} D \otimes_{K_0} \mathbf{B}_{\text{cris}}$  compatible with the Frobenius on both sides. Moreover, it maps  $(\mathbf{B}_{\text{cris}}^+)^{\oplus h}$  onto the preimage of  $\text{Fil}_\mu^1 D_C$  under the map  $\text{id} \otimes \theta : D \otimes_{K_0} \mathbf{B}_{\text{cris}}^+ \rightarrow D \otimes_{K_0} C$ . This means that  $((D, \varphi_D, \text{Fil}_\mu^\bullet D_L) \otimes \mathbb{1}(-1))^\vee$  is admissible in the sense of Faltings's [Fal10, Definition 1]. Note that Faltings uses contravariant Dieudonné modules. By [Fal10, Theorems 9 and 14],  $\text{Fil}_\mu^1 D_L = \mathbb{D}(\eta^{-1})(\text{Lie } X^\vee)_L^\vee$  for a  $p$ -divisible group  $X$  over  $\mathcal{O}_L$  and a quasi-isogeny  $\eta : \mathbf{X}_{\mathcal{O}_L/(p)} \rightarrow X_{\mathcal{O}_L/(p)}$ , hence  $\mu$  lies in the image of  $\check{\pi}^{\text{an}}$  as desired.

(b) This is proved by de Jong [dJ95]. Indeed by [dJ95, Proposition 6.2] the rational Tate module  $V_p X_{\check{\mathcal{G}}^{\text{an}}}$  is a local system of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{G}}^{\text{an}}$ . In order that it defines a local system  $\mathcal{V}$  on  $\check{\mathcal{F}}_b^0$  it suffices by [dJ95, Definition 4.1] to show that

- (i)  $\check{\pi} : \check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{F}}_b^0$  is a covering for the étale topology.
- (ii) There is a descent datum  $\tilde{\psi} : pr_1^* V_p X_{\check{\mathcal{G}}^{\text{an}}} \xrightarrow{\sim} pr_2^* V_p X_{\check{\mathcal{G}}^{\text{an}}}$  over  $\check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}}$  where  $pr_i : \check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{G}}^{\text{an}}$  is the projection onto the  $i$ -th factor, such that  $\tilde{\psi}$  satisfies the cocycle condition on  $\check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}}$ .

Statement (i) was proved by Fargues [Far04, Lemma 2.3.24] and also follows from Faltings result [Fal10, Theorem 14] which says that  $\check{\mathcal{F}}_b^0$  has an open covering  $\check{\mathcal{F}}_b^0 = \bigcup_i U_i$  such that the morphism  $\coprod_i U_i \rightarrow \check{\mathcal{F}}_b^0$  factors through  $\check{\pi} : \check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{F}}_b^0$ .

For (ii) consider the universal filtered isocrystal  $\underline{\mathcal{D}} = (V_{K_0}, b\varphi, \text{Fil}^\bullet)$  over  $\check{\mathcal{F}}_b^0$  and its canonical descent datum  $\text{id} : pr_1^* \check{\pi}^* \underline{\mathcal{D}} = pr_2^* \check{\pi}^* \underline{\mathcal{D}}$  over  $\check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}}$ . De Jong considers  $pr_i^* X_{\check{\mathcal{G}}^{\text{an}}}$  as an object of the stack  $BT_{\mathbb{Q}, \check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}}}^{\text{rig}}$  of  $p$ -divisible groups up to isogeny over  $\check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}}$ . The functor from this stack to filtered isocrystals sends  $pr_i^* X_{\check{\mathcal{G}}^{\text{an}}}$  to  $pr_i^* \check{\pi}^* \underline{\mathcal{D}}$ . Since this functor is fully faithful by [dJ95, Proposition 6.6] one obtains a descent datum  $\psi : pr_1^* X_{\check{\mathcal{G}}^{\text{an}}} \xrightarrow{\sim} pr_2^* X_{\check{\mathcal{G}}^{\text{an}}}$  in  $BT_{\mathbb{Q}, \check{\mathcal{G}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} \check{\mathcal{G}}^{\text{an}}}^{\text{rig}}$ . One now takes  $\tilde{\psi} = V_p \psi : pr_1^* V_p X_{\check{\mathcal{G}}^{\text{an}}} \xrightarrow{\sim} pr_2^* V_p X_{\check{\mathcal{G}}^{\text{an}}}$ . This yields the local system  $\mathcal{V}$  on  $\check{\mathcal{F}}_b^0$ .

The local system  $\mathcal{V}$  induces the desired tensor functor  $\underline{\mathcal{V}}$  because  $V$  is a tensor generator of  $\text{Rep}_{\mathbb{Q}_p} \text{GL}(V)$ . Let us give some more details. We do not know whether  $\check{\mathcal{F}}_b^0$  is connected, although we expect this to hold; see Conjecture 5.5. Therefore we consider each connected component  $Y$  of it separately and fix a geometric base point  $\bar{\mu}$  of  $Y$ . Then  $\mathcal{V}$  induces a representation

$$\pi_1^{\text{ét}}(Y, \bar{\mu}) \longrightarrow \text{GL}(\mathcal{V}_{\bar{\mu}})(\mathbb{Q}_p)$$

by Proposition 1.2. We use the notation introduced before Corollary 1.5. Since  $\dim V = \dim \mathcal{V}_{\bar{\mu}}$  and  $\text{H}^1(\mathbb{Q}_p, \text{GL}(V)) = (0)$  by Hilbert 90, we can take  $\tilde{\omega} = \omega_0$  and  $\tilde{G} = \text{GL}(V) \cong \text{GL}(\mathcal{V}_{\bar{\mu}})$ . This defines a continuous group homomorphism  $\pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p)$  and via Corollary 1.5 a tensor functor  $\underline{\mathcal{V}} : \text{Rep}_{\mathbb{Q}_p} \text{GL}(V) \rightarrow \mathbb{Q}_p\text{-Loc}_Y$  satisfying (1.9). Tracing through the

proof of Corollary 1.5 we see that its composition with the forgetful fiber functor  $\omega_{\bar{\mu}}$  satisfies  $\omega_{\bar{\mu}} \circ \underline{\mathcal{V}} \cong \tilde{\omega} = \omega_0$ . Hence  $\omega_{\bar{\mu}} \circ \underline{\mathcal{V}}(V) \cong V \cong \mathcal{V}_{\bar{\mu}}$  and  $\underline{\mathcal{V}}(V) \cong \mathcal{V}$ . Moreover, for every  $\mu \in \check{\mathcal{F}}_b^0(L)$  with  $L/K_0$  finite the  $\text{Gal}(L^{\text{alg}}/L)$ -representations satisfy

$$\underline{\mathcal{V}}(V)_{\mu} \cong V_p X_{\mathcal{O}_L} \cong V_{\text{cris}}(\mathbb{D}(X_{\mathcal{O}_L}), (\text{Lie } X_{\mathcal{O}_L}^{\vee})^{\vee}) \cong V_{\text{cris}}(\underline{D}_{b,\mu}(V)).$$

This holds on all connected components  $Y$  and therefore gives the tensor functor  $\underline{\mathcal{V}}$  satisfying (1.8) of Conjecture 1.3 on all of  $\check{\mathcal{F}}_b^0$  as claimed.

(c) We fix a geometric base point  $\bar{\mu}$  of  $\check{\mathcal{F}}_b^0$  and consider the canonical family of morphisms  $f_{\tilde{K}} : \check{\mathcal{G}}_{\tilde{K}}^{\text{an}} \rightarrow \check{\mathcal{E}}_{\tilde{K}}$  which sends  $\alpha \tilde{K}$  for  $\alpha : \tilde{V} \xrightarrow{\sim} (V_p X_{\check{\mathcal{G}}^{\text{an}}})_{\bar{\mu}}$  to the  $\tilde{K}$ -residue class  $\beta \tilde{K}$  of tensor isomorphisms  $\beta : \tilde{\omega} \xrightarrow{\sim} \omega_{\bar{\mu}} \circ \underline{\mathcal{V}}$  induced by  $\beta_V := \alpha : \tilde{\omega}(V) = \tilde{V} \xrightarrow{\sim} (V_p X_{\check{\mathcal{G}}^{\text{an}}})_{\bar{\mu}} = \omega_{\bar{\mu}} \circ \underline{\mathcal{V}}(V)$  via the fact that  $V$  is a tensor generator of  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$ . Again  $f_{\tilde{K}}$  is independent of the choice of  $\bar{\mu}$  by [dJ95, Theorem 2.9]. By construction the family  $(f_{\tilde{K}})_{\tilde{K}}$  is equivariant for the Hecke action of  $\tilde{G}(\mathbb{Q}_p)$  on both towers. For any algebraically closed complete extension  $C$  of  $\check{E}$  the morphism  $f_{\tilde{K}}$  is bijective on  $C$ -valued points because the fibers of  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}(C)$  and  $\check{\mathcal{E}}_{\tilde{K}}(C)$  over a fixed  $C$ -valued point of  $\check{\mathcal{F}}_b^0$  are both isomorphic to  $\tilde{G}(\mathbb{Q}_p)/\tilde{K}$  by [RZ96, Proposition 5.37] and Remark 1.6. Hence  $f_{\tilde{K}}$  is quasi-finite by [Beh93, Proposition 3.1.4]. Since  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$  and  $\check{\mathcal{E}}_{\tilde{K}}$  are étale over  $\check{\mathcal{F}}_b^0$  the morphisms  $f_{\tilde{K}}$  are étale by [Beh93, Corollary 3.3.9] and hence isomorphisms by Proposition A.4.

The statement about  $\check{\mathcal{G}}^{\text{an}}$  comes from the identification  $\check{\mathcal{G}}^{\text{an}} \cong \check{\mathcal{G}}_{\text{GL}(\tilde{\Lambda})}^{\text{an}}$  described before the theorem and the fact that  $\check{\mathcal{E}}_{\text{GL}(\tilde{\Lambda})}$  is the space of  $\mathbb{Z}_p$ -lattices inside  $\mathcal{V}$ .  $\square$

**Corollary 6.4.** *Let  $G = \text{GL}(V)$ , let  $\tilde{b} \in G(K_0)$ , and let  $\{\tilde{\mu}\}$  be a conjugacy class of cocharacters  $\mathbb{G}_m \rightarrow G$  such that  $V$  satisfies (5.1) for some integer  $\tilde{n}$ . Then there is a tensor functor  $\tilde{\mathcal{V}}$  from  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  to the category  $\mathbb{Q}_p\text{-Loc}_{\check{\mathcal{F}}_b^0}$  of local systems of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$  satisfying (1.8) of Conjecture 1.3.*

*Proof.* As at the beginning of this section we set  $b := p^{1-\tilde{n}} \cdot \tilde{b} \in G(K_0)$  and define  $\mu : \mathbb{G}_m \rightarrow G$  by  $\mu(z) = z^{1-\tilde{n}} \cdot \tilde{\mu}(z)$  to obtain identifications  $\underline{D}_{b,\mu}(V) = \underline{D}_{\tilde{b},\tilde{\mu}}(V) \otimes \mathbb{1}(1-\tilde{n})$  and  $\check{\mathcal{F}}_b^0 \xrightarrow{\sim} \check{\mathcal{F}}_{\tilde{b}}^0$ ,  $\tilde{\mu} \mapsto \mu$ . Then Theorem 6.3(b) yields a tensor functor  $\underline{\mathcal{V}}$ , which for every  $L$ -valued point  $\tilde{\mu} \in \check{\mathcal{F}}_{\tilde{b}}^0$  for  $L/K_0$  finite satisfies

$$\underline{\mathcal{V}}(V)_{\mu} \cong V_{\text{cris}}(\underline{D}_{b,\mu}(V)) = V_{\text{cris}}(\underline{D}_{\tilde{b},\tilde{\mu}}(V)) \otimes \mathbb{Q}_p(\tilde{n}-1)$$

as  $\text{Gal}(L^{\text{alg}}/L)$ -representations because  $V_{\text{cris}}(\mathbb{1}(1)) = \mathbb{Q}_p(-1)$ . Thus it remains to twist  $\underline{\mathcal{V}}$  with the cyclotomic character. In concrete terms this means the following. On each connected component  $Y$  of  $\check{\mathcal{F}}_b^0$  with some geometric base point  $\bar{\mu}$  the tensor functor  $\underline{\mathcal{V}}$  corresponds to a representation  $\rho' : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p) \cong \text{GL}_h(\mathbb{Q}_p)$ . Consider the representation  $\tilde{\rho}' : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \text{GL}_h(\mathbb{Q}_p)$  with  $\tilde{\rho}'(\gamma) := \chi_{\text{cyc}}(\gamma)^{1-\tilde{n}} \cdot \rho'(\gamma)$  where  $\chi_{\text{cyc}} : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \text{Gal}(\check{E}^{\text{alg}}/\check{E}) \xrightarrow{\chi_{\text{cyc}}} \mathbb{Z}_p^{\times}$  is the cyclotomic character of the base field  $\check{E}$ . Let  $\tilde{\mathcal{V}}$  be the tensor functor from  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  to  $\mathbb{Q}_p\text{-Loc}_{\check{\mathcal{F}}_b^0}$  associated with  $\tilde{\rho}'$  by Corollary 1.5. It is independent of the base point  $\bar{\mu}$  and hence exists on all of  $\check{\mathcal{F}}_b^0$ . Then  $\tilde{\mathcal{V}}(V)_{\tilde{\mu}} \cong V_{\text{cris}}(\underline{D}_{\tilde{b},\tilde{\mu}}(V))$  as  $\text{Gal}(L^{\text{alg}}/L)$ -representations for all  $\tilde{\mu}$  and this proves the corollary.  $\square$

Finally we can give the

*Proof of Theorem 5.8.* Let  $\rho : G \hookrightarrow \mathrm{GL}(V) =: G'$  be a representation satisfying (5.1) and consider the induced embedding  $\check{\mathcal{F}} \hookrightarrow \check{\mathcal{F}}' := \mathcal{F}lag(V)$ . By Proposition 5.2 we have  $\check{\mathcal{F}}_b^0 = \check{\mathcal{F}}^{\mathrm{an}} \cap \check{\mathcal{F}}_b'^0$ . Now the local system  $\check{\mathcal{Y}}(V)$  of Corollary 6.4 on  $\check{\mathcal{F}}_b'^0$  pulls back to a local system of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$  which has the required property.  $\square$

**Conjecture 6.5.** *The open subspace  $\check{\mathcal{F}}_b^0$  from Corollary 6.4 is the largest open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^a$  of  $\check{\mathcal{F}}_b^{wa}$  on which a tensor functor  $\underline{\mathcal{Y}} : \underline{\mathrm{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-}\underline{\mathrm{Loc}}_{\check{\mathcal{F}}_b^a}$  exists which satisfies (1.8) of Conjecture 1.3.*

In particular, we expect that  $\check{\mathcal{F}}_b^0$  and  $\underline{\mathcal{Y}}$  from Corollary 6.4 solve the problem posed in Conjecture 1.3 for  $G = \mathrm{GL}(V)$ . Clearly the conjecture follows from Conjecture 5.10 because  $\underline{\mathcal{Y}}(V)$  is a local system as in Conjecture 5.10. We give some evidence for both conjectures in Section 9.

## 7 PEL period morphisms

Rapoport and Zink also consider period morphisms in the PEL situation with parahoric level structure. They fix data as follows:

### Case EL:

- Let  $B$  be a finite semi-simple algebra over  $\mathbb{Q}_p$ .
- Let  $\mathcal{O}_B$  be a maximal order in  $B$ .
- Let  $V$  be a finite dimensional  $B$ -module.
- Let  $G = \mathrm{GL}_B(V)$  as an algebraic group over  $\mathbb{Q}_p$ .

### Case PEL:

- Let  $B, \mathcal{O}_B, V$  be as in case EL.
- Let  $(\cdot, \cdot)$  be a nondegenerate alternating  $\mathbb{Q}_p$ -bilinear form on  $V$ .
- $*$  :  $a \mapsto a^*$  an involution of  $B$  which satisfies

$$(av, w) = (v, a^*w) \quad \text{for all } v, w \in V.$$

Let  $G$  be the algebraic group over  $\mathbb{Q}_p$  whose points with values in a  $\mathbb{Q}_p$ -algebra  $R$  are  $G(R) = \{ g \in \mathrm{GL}_B(V \otimes R) : \exists c(g) \in R \text{ with } (gv, gw) = c(g)(v, w) \ \forall v, w \in V \}$ .

In addition they fix an element  $b \in G(K_0)$ , a conjugacy class of cocharacters  $\mu : \mathbb{G}_m \rightarrow G$  and a multichain  $\mathcal{L}$  of  $\mathcal{O}_B$ -lattices in  $V$  [RZ96, Definition 3.4]. They assume that the representation  $V$  of  $G$  satisfies our condition (5.1) with  $n = 1$ . In case PEL they assume in addition that the character  $c : G \rightarrow \mathbb{G}_m$  pairs with the slope quasi-cocharacter  $\nu \in \mathrm{Hom}_{K_0}(\mathbb{G}_m, G) \otimes_{\mathbb{Z}} \mathbb{Q}$  to 1, and that  $\mathcal{L}$  is self-dual [RZ96, Definition 3.13].

Under these assumptions there is a  $p$ -divisible group  $\mathbf{X}$  over  $\mathbb{F}_p^{\mathrm{alg}}$  whose covariant Dieudonné module is  $(V_{K_0}, b \cdot \varphi)$ . In particular  $\mathbf{X}$  is equipped with an action  $i_{\mathbf{X}} : B \rightarrow \mathrm{End}(\mathbf{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  and, in case PEL a polarization  $\lambda : \mathbf{X} \rightarrow \mathbf{X}^\vee$  with  $\lambda^\vee = -\lambda$ , which is uniquely determined up to multiplication by an element of  $\mathbb{Q}_p^\times$ ; see [RZ96, 3.20]. Let  $E$  be the Shimura field, that is the field of definition of the conjugacy class of  $\mu$ , and let  $\check{E} = E \cdot K_0$ . Rapoport and Zink consider the following moduli problem.

**Definition 7.1** ([RZ96, Definition 3.21]). Let  $\check{\mathcal{G}} : \mathcal{N}ilp_{\mathcal{O}_{\check{E}}} \rightarrow \mathbf{Sets}$  be the functor which assigns to  $S \in \mathcal{N}ilp_{\mathcal{O}_{\check{E}}}$  the following data up to isomorphism.

- (a) For each lattice  $\Lambda \in \mathcal{L}$  a  $p$ -divisible group  $X_\Lambda$  over  $S$ , with an  $\mathcal{O}_B$ -action,  $i_\Lambda : \mathcal{O}_B \rightarrow \text{End } X_\Lambda$ .
- (b) For each lattice  $\Lambda \in \mathcal{L}$  a quasi-isogeny  $\eta_\Lambda : \mathbf{X}_{\bar{S}} \rightarrow X_{\Lambda, \bar{S}}$  which commutes with the action of  $\mathcal{O}_B$ . (Here again  $\bar{S}$  is the closed subscheme of  $S$  defined by the ideal  $p\mathcal{O}_S$ .)

such that certain conditions (i)–(v) are satisfied for which we refer to [RZ96, Definition 3.21 and the discussion on pp. 82–88].

**Proposition 7.2** ([RZ96, Theorem 3.25]). *The functor  $\check{\mathcal{G}} : \text{Nilp}_{\mathcal{O}_{\check{E}}} \rightarrow \text{Sets}$  is representable by a formal scheme locally formally of finite type over  $\mathcal{O}_{\check{E}}$ .*

The period morphism  $\check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{F}}_b^{wa}$  is defined as follows. Fix a  $\Lambda \in \mathcal{L}$ , let  $X_\Lambda$  be the universal  $p$ -divisible group over  $\check{\mathcal{G}}$ , and consider the exact sequence

$$0 \longrightarrow (\text{Lie } X_\Lambda^\vee)^\vee \longrightarrow \mathbb{D}(X_\Lambda)_{\check{\mathcal{G}}} \longrightarrow \text{Lie } X_\Lambda \longrightarrow 0.$$

Then the quasi-isogeny  $\eta_\Lambda$  induces an isomorphism

$$\mathbb{D}(\eta_\Lambda)_{\check{\mathcal{G}}^{\text{an}}} : V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\check{\mathcal{G}}^{\text{an}}} = \mathbb{D}(\mathbf{X})_{\check{\mathcal{G}}^{\text{an}}} \xrightarrow{\sim} \mathbb{D}(X_\Lambda)_{\check{\mathcal{G}}^{\text{an}}}$$

that makes  $\mathbb{D}(\eta_\Lambda)_{\check{\mathcal{G}}^{\text{an}}}^{-1} (\text{Lie } X^\vee)_{\check{\mathcal{G}}^{\text{an}}}^\vee$  into a family over  $\check{\mathcal{G}}^{\text{an}}$  of  $B$ -invariant subspaces of  $V$ , which in case PEL are totally isotropic. These subspaces are classified by  $\check{\mathcal{F}}^{\text{an}}$  and this defines the *period morphism*  $\check{\pi} : \check{\mathcal{G}}^{\text{an}} \rightarrow \check{\mathcal{F}}^{\text{an}}$ . The whole construction does not depend on the choice of  $\Lambda \in \mathcal{L}$ ; see [RZ96, 5.16]. The period morphism factors through  $\check{\mathcal{F}}_b^{wa}$  and by the same argument as in Proposition 6.2 it even factors through  $\check{\mathcal{F}}_b^0$ . In Theorem 7.4 below we will determine the image of the period morphism  $\check{\pi}$ .

Recall the cohomology class

$$cl(b, \mu) = \kappa(b) - \mu^\# \in H^1(\mathbb{Q}_p, G)$$

and the associated fiber functor  $\tilde{\omega} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow (\mathbb{Q}_p\text{-vector spaces})$  and inner form  $\tilde{G}$  of  $G$  described before Corollary 1.5. Let  $\tilde{V} := \tilde{\omega}(V)$ . Since  $B$  acts as endomorphisms of the representation  $V$  of  $G$ , also  $\tilde{V}$  is a  $B$ -module. In the PEL case  $\tilde{V}$  is equipped with a non-degenerate alternating  $\mathbb{Q}_p$ -bilinear form  $(\cdot, \cdot)^\sim$  such that

$$(a\tilde{v}, \tilde{w})^\sim = (\tilde{v}, a^*\tilde{w})^\sim \quad \text{for all } \tilde{v}, \tilde{w} \in \tilde{V} \text{ and all } a \in B. \quad (7.1)$$

Namely, the original form  $(\cdot, \cdot) : V \otimes_{\mathbb{Q}_p} V \rightarrow \mathbb{Q}_p$  is a morphism from the  $G$ -representation  $V \otimes_{\mathbb{Q}_p} V$  to the  $G$ -representation  $\mathbb{Q}_p$  on which  $G$  acts through the character  $c$ . We let  $(\cdot, \cdot)^\sim$  be the image of  $(\cdot, \cdot)$  under  $\tilde{\omega}$ . Formula (7.1), being an equality of morphisms  $\tilde{V} \otimes_{\mathbb{Q}_p} \tilde{V} \rightarrow \mathbb{Q}_p$ , is the image under  $\tilde{\omega}$  of the corresponding equality for  $V$ . By construction of  $\tilde{G}$  we have  $\tilde{G} \cong \text{Aut}^\otimes(\tilde{\omega})$  and  $\underline{\text{Rep}}_{\mathbb{Q}_p} G \cong \underline{\text{Rep}}_{\mathbb{Q}_p} \tilde{G}$ ; see [DM82, Proposition 2.8 and Theorem 2.11]. Applying  $\tilde{\omega}$  to the character  $c : G \rightarrow \mathbb{G}_m$  we obtain another character which we view as a  $\tilde{G}$ -representation  $\tilde{c} : \tilde{G} \rightarrow \mathbb{G}_m$ . In this way we get identifications of algebraic groups over  $\mathbb{Q}_p$

$$\begin{aligned} \tilde{G} &= \text{GL}_B(\tilde{V}) && \text{in case EL,} && (7.2) \\ \tilde{G} &= \{ \tilde{g} \in \text{GL}_B(\tilde{V}) : (\tilde{g}\tilde{v}, \tilde{g}\tilde{w})^\sim = \tilde{c}(\tilde{g})(\tilde{v}, \tilde{w})^\sim \ \forall \tilde{v}, \tilde{w} \in \tilde{V} \} && \text{in case PEL.} \end{aligned}$$

Consider a decomposition of  $B$  into a product  $\prod_{i=1}^r B_i$  of simple algebras  $B_i \cong M_{n_i}(D_i)$ , which are matrix algebras over division algebras  $D_i$ , in such a way that  $\mathcal{O}_B \cong M_{n_i}(\mathcal{O}_{D_i})$ . Then any  $\mathcal{O}_B$ -module  $\Lambda$  decomposes accordingly  $\Lambda \cong \bigoplus_{i=1}^r \Lambda_i$ .

**Definition 7.3.** In the EL case a *multichain* of type  $(\mathcal{L})$  of  $\mathcal{O}_B$ -lattices in  $\tilde{V}$  is a functor  $\sim : \Lambda \mapsto \tilde{\Lambda}$  from  $\mathcal{L}$  to the category of  $\mathcal{O}_B$ -lattices in  $\tilde{V}$  such that

- (ii) If  $\Lambda \subset \Lambda'$  then  $\tilde{\Lambda} \subset \tilde{\Lambda}'$  and  $[\tilde{\Lambda}'_i : \tilde{\Lambda}_i] = [\Lambda'_i : \Lambda_i]$  for  $i = 1, \dots, r$ .
- (iii) For any  $a \in B$  with  $a^{-1}\mathcal{O}_B a = \mathcal{O}_B$  we have  $\tilde{a}\tilde{\Lambda} = a \cdot \tilde{\Lambda}$ .

In the PEL case we speak of a *polarized multichain* of type  $(\mathcal{L})$  and we require in addition that there is a unit  $\tilde{\ell} \in \mathbb{Q}_p^\times$  with

- (v)  $\tilde{\ell} \cdot (\tilde{\Lambda}^\vee) = (\tilde{\Lambda})^\vee := \{ \tilde{v} \in \tilde{V} : (\tilde{v}, \tilde{w})^\sim \in \mathbb{Z}_p \text{ for all } \tilde{w} \in \tilde{\Lambda} \}$  for every  $\Lambda \in \mathcal{L}$ .

The unusual numbering of our conditions is chosen because the items of our definition correspond to the respective items of [RZ96, Definition 3.21].

For the next theorem again recall from Remark 1.6 that with each tensor functor  $\underline{\mathcal{V}} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-}\underline{\text{Loc}}_{\check{\mathcal{F}}_b^0}$  one associates a tower of étale covering spaces  $\check{\mathcal{E}}_{\tilde{K}}$  of  $\check{\mathcal{F}}_b^0$  for  $\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)$  compact open, and a tower of covering spaces  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}}$  of  $\check{\mathcal{G}}^{\text{an}}$ ; see Rapoport and Zink [RZ96, 5.32–5.39] and our discussion before Theorem 6.3.

**Theorem 7.4.** *Assume that  $\check{\mathcal{G}}^{\text{an}} \neq \emptyset$ .*

- (a) *Then the image  $\check{\pi}(\check{\mathcal{G}}^{\text{an}})$  of the EL or PEL period morphism  $\check{\pi}$  equals the open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}^{\text{an}}$  from Theorem 5.6.*
- (b) *Fix any lattice  $\Lambda \in \mathcal{L}$ . Then the rational Tate module  $V_p X_{\Lambda, \check{\mathcal{G}}^{\text{an}}}$  of the universal  $p$ -divisible group  $X_{\Lambda, \check{\mathcal{G}}^{\text{an}}}$  over  $\check{\mathcal{G}}^{\text{an}}$  descends to a local system  $\mathcal{V}$  of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$  which is independent of  $\Lambda$  up to canonical isomorphism. This induces a tensor functor  $\underline{\mathcal{V}}$  from  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  to the category  $\mathbb{Q}_p\text{-}\underline{\text{Loc}}_{\check{\mathcal{F}}_b^0}$  of local systems of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$  satisfying (1.8) of Conjecture 1.3 and  $\underline{\mathcal{V}}(V) \cong \mathcal{V}$ .*
- (c) *The tower of  $\check{E}$ -analytic spaces  $(\check{\mathcal{G}}_{\tilde{K}}^{\text{an}})_{\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)}$  is canonically isomorphic over  $\check{\mathcal{F}}_b^0$  in a Hecke equivariant way to the tower of étale covering spaces  $(\check{\mathcal{E}}_{\tilde{K}})_{\tilde{K} \subset \tilde{G}(\mathbb{Q}_p)}$  of  $\check{\mathcal{F}}_b^0$  associated with the tensor functor  $\underline{\mathcal{V}}$ . In particular,  $\check{\mathcal{G}}^{\text{an}}$  is (non-canonically) isomorphic to the space of (polarized in case PEL) multichains of type  $(\mathcal{L})$  inside the local system  $\mathcal{V}$  (which are defined as in Definition 7.3 using Proposition 1.2).*

**Remark 7.5.** For each connected component  $Y$  of  $\check{\mathcal{F}}_b^0$  and geometric base point  $\bar{\mu}$  of  $Y$  the tensor functor  $\underline{\mathcal{V}}$  corresponds to a representation  $\rho' : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p)$  by Corollary 1.5. In the course of the proof we show that the composition  $\tilde{c} \circ \rho'$  with the multiplier  $\tilde{c}$  of  $\tilde{G}$  equals the cyclotomic character  $\pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \text{Gal}(\check{E}^{\text{alg}}/\check{E}) \xrightarrow{\chi_{\text{cyc}}} \mathbb{Z}_p^\times$  of the base field  $\check{E}$ .

Also (c) implies that  $\check{\mathcal{G}}_{\tilde{K}}^{\text{an}} \times_{\check{\mathcal{F}}_b^0} Y$  is the étale covering space of  $Y$  corresponding to the  $\pi_1^{\text{ét}}(Y, \bar{\mu})$ -set  $\tilde{G}(\mathbb{Q}_p)/\tilde{K}$ .

Before proving the theorem let us mention the following

**Conjecture 7.6.** *The open subspace  $\check{\mathcal{F}}_b^0$  is the largest open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^a$  of  $\check{\mathcal{F}}_b^{wa}$  on which a tensor functor  $\underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-}\underline{\text{Loc}}_{\check{\mathcal{F}}_b^a}$  exists which satisfies (1.8) of Conjecture 1.3.*

In particular, we expect that  $\check{\mathcal{F}}_b^0$  and  $\check{\mathcal{V}}$  from Theorem 7.4 (b) solve the problem posed in Conjecture 1.3. Again the conjecture follows from Conjecture 5.10 because  $\check{\mathcal{V}}(V)$  is a local system as in Conjecture 5.10. We give some evidence for both conjectures in Section 9. We now turn to the

*Proof of Theorem 7.4.* (a) 1. Since  $\check{\mathcal{G}}^{\text{an}} \neq \emptyset$  there is a finite field extension  $L/\check{E}$  and a point  $x_0 \in \check{\mathcal{G}}(\mathcal{O}_L)$  corresponding to an  $\mathcal{L}$ -set of  $p$ -divisible groups with  $\mathcal{O}_B$ -action  $\{X_\Lambda, i_\Lambda\}_{\Lambda \in \mathcal{L}}$  and  $\mathcal{O}_B$ -equivariant quasi-isogenies  $\eta_\Lambda : \mathbf{X}_{\mathcal{O}_L/(p)} \rightarrow X_{\Lambda, \mathcal{O}_L/(p)}$  satisfying conditions (i)–(v) of [RZ96, Definition 3.21]. In particular  $\eta_{\Lambda'} \eta_\Lambda^{-1}$  lifts to a quasi-isogeny  $\eta_{\Lambda', \Lambda} : X_\Lambda \rightarrow X_{\Lambda'}$  over  $\mathcal{O}_L$  for all  $\Lambda, \Lambda' \in \mathcal{L}$ . Let  $\mu_\Lambda = \check{\pi}(x_0) \in \check{\mathcal{F}}_b^0$  be the image of  $x_0$ . By Wintenberger [Win97, Corollary to Proposition 4.5.3] the rational Tate modules  $V_p X_{\Lambda, L^{\text{alg}}}$  are all  $B$ -isomorphic to  $\tilde{V}$  via  $B$ -isomorphisms  $\delta_\Lambda : V_p X_\Lambda \xrightarrow{\sim} \tilde{V}$  satisfying  $\delta_\Lambda = \delta_{\Lambda'} \circ V_p(\eta_{\Lambda', \Lambda})$ . We claim that  $\delta_\Lambda(T_p X_\Lambda) =: \tilde{\Lambda} \subset \tilde{V}$  forms a (polarized) multichain of type  $(\mathcal{L})$  of  $\mathcal{O}_B$ -lattices in  $\tilde{V}$ . Indeed the decomposition  $\mathcal{O}_B \cong \prod_{i=1}^r M_{n_i}(\mathcal{O}_{D_i})$  induces a decomposition  $X_\Lambda \cong \prod_{i=1}^r X_{\Lambda_i}$  of  $p$ -divisible groups. If  $\Lambda \subset \Lambda'$  then [RZ96, Definition 3.21(ii) and (ii bis) from p. 88] implies that  $\eta_{\Lambda', \Lambda}$  is an isogeny whose  $i$ -th composition factor  $\eta_{\Lambda'_i, \Lambda_i} : X_{\Lambda_i} \rightarrow X_{\Lambda'_i}$  has height  $\log_p[\Lambda'_i : \Lambda_i]$ . Hence  $\tilde{\Lambda} \subset \tilde{\Lambda}'$  and  $[\tilde{\Lambda}'_i : \tilde{\Lambda}_i] = p^{\text{ht}(\eta_{\Lambda'_i, \Lambda_i})} = [\Lambda'_i : \Lambda_i]$  proving (ii).

If  $a \in B^\times$  satisfies  $a^{-1} \mathcal{O}_B a = \mathcal{O}_B$  then [RZ96, Definition 3.21(iii)] says that  $\eta_{a\Lambda, \Lambda} = \theta_{a, \Lambda} \circ i_\Lambda(a)^{-1}$  for an isomorphism  $\theta_{a, \Lambda} : X_\Lambda \rightarrow X_{a\Lambda}$  of  $p$ -divisible groups. Hence (iii) follows from

$$a\tilde{\Lambda} = \delta_\Lambda V_p(\eta_{\Lambda, a\Lambda})(T_p X_{a\Lambda}) = \delta_\Lambda V_p(i_\Lambda(a) \theta_{a, \Lambda}^{-1})(T_p X_{a\Lambda}) = a \cdot \delta_\Lambda(T_p X_\Lambda) = a \cdot \tilde{\Lambda}.$$

In case PEL there exists by [RZ96, Definition 3.21(v)] a constant  $\ell \in \mathbb{Q}_p^\times$  such that for all  $\Lambda \in \mathcal{L}$  the quasi-isogeny  $(\eta_{\Lambda^\vee}^{-1})^{-1} \circ \ell \lambda \circ \eta_\Lambda^{-1}$

$$\begin{array}{ccc} X_{\Lambda, \mathcal{O}_L/(p)} & \longrightarrow & (X_{\Lambda^\vee, \mathcal{O}_L/(p)})^\vee \\ \eta_\Lambda \uparrow & & \downarrow (\eta_{\Lambda^\vee})^\vee \\ \mathbf{X}_{\mathcal{O}_L/(p)} & \xrightarrow{\ell \lambda} & \mathbf{X}_{\mathcal{O}_L/(p)}^\vee \end{array}$$

lifts to an isomorphism  $p_\Lambda : X_\Lambda \xrightarrow{\sim} (X_{\Lambda^\vee})^\vee$ . Here the polarization  $\ell \lambda : \mathbf{X} \rightarrow \mathbf{X}^\vee$  comes from the alternating form  $(\cdot, \cdot)$  on  $V$ . That is  $(\cdot, \cdot)$  on  $V \otimes_{\mathbb{Q}_p} K_0 = \mathbb{D}(\mathbf{X})_{K_0}$  coincides up to multiplication with a unit in  $\mathbb{Q}_p^\times$  with the pairing

$$\mathbb{D}(\mathbf{X})_{K_0} \otimes_{K_0} \mathbb{D}(\mathbf{X})_{K_0} \xrightarrow{\mathbb{D}(\ell \lambda) \otimes \text{id}} \mathbb{D}(\mathbf{X}^\vee)_{K_0} \otimes_{K_0} \mathbb{D}(\mathbf{X})_{K_0} \xrightarrow{\mathbb{D}(\psi_{\mathbf{X}})_{K_0}} \mathbb{D}(\mathbb{G}_m)_{K_0} = K_0$$

induced from the pairing  $\psi_{\mathbf{X}} : \mathbf{X}^\vee \times \mathbf{X} \rightarrow \mathbb{G}_m$ . Under the identification  $\delta_\Lambda : V_p X_\Lambda \xrightarrow{\sim} \tilde{V}$  the isomorphism  $p_\Lambda$  induces an alternating form

$$\tilde{V} \otimes_{\mathbb{Q}_p} \tilde{V} \xrightarrow{V_p(p_{\Lambda^\vee} \circ \eta_{\Lambda^\vee, \Lambda}) \delta_\Lambda^{-1} \otimes \delta_\Lambda^{-1}} V_p X_{\Lambda^\vee} \otimes_{\mathbb{Q}_p} V_p X_\Lambda \xrightarrow{V_p \psi_\Lambda} V_p \mathbb{G}_{m, L^{\text{alg}}} = \mathbb{Q}_p \quad (7.3)$$

which is independent of  $\Lambda$ . Up to multiplication with a unit  $\tilde{\ell} \in \mathbb{Q}_p^\times$  this alternating form equals  $(\cdot, \cdot)^\sim$ , the image of  $(\cdot, \cdot)$  under the fiber functor  $\tilde{\omega}$  by Wintenberger [Win97,

Corollary to Proposition 4.5.3]. Hence  $(\tilde{v})$  follows from

$$\begin{aligned}
\tilde{\Lambda}^\vee &:= \delta_\Lambda \circ V_p(\eta_{\Lambda, \Lambda^\vee})(T_p X_{\Lambda^\vee}) \\
&= \delta_\Lambda \circ V_p(\eta_{\Lambda, \Lambda^\vee} \circ p_{\Lambda^\vee}^{-1})(T_p X_\Lambda^\vee) \\
&= \{ \tilde{v} \in \tilde{V} : \tilde{\ell} \cdot (\tilde{v}, \tilde{w}) \sim \in \mathbb{Z}_p \text{ for all } \tilde{w} \in \delta_\Lambda(T_p X_\Lambda) = \tilde{\Lambda} \} \\
&= \tilde{\ell}^{-1}(\tilde{\Lambda})^\vee.
\end{aligned}$$

2. Now in addition to  $\mu_0$  let  $\mu \in \check{\mathcal{F}}_b^0$  be any point. Consider the canonical representation  $\rho : G \hookrightarrow \mathrm{GL}(V) =: G'$  and the induced embedding  $\check{\mathcal{F}}_b^0 \hookrightarrow \check{\mathcal{F}}_{\rho(b)}^0$ ,  $\mu \mapsto \rho \circ \mu$  into the corresponding open subspace  $\check{\mathcal{F}}_{\rho(b)}^0$  for  $G'$ . The properties of the latter were discussed in the previous section. Moreover,  $\check{\mathcal{F}}_b^0 = \check{\mathcal{F}}^{\mathrm{an}} \cap \check{\mathcal{F}}_{\rho(b)}^0$  by Proposition 5.2. In addition we denote the formal scheme from Proposition 6.1 in this section by  $\check{\mathcal{G}}'$ . By Theorem 6.3 and Faltings [Fal10, Theorem 14] there is an admissible affine formal scheme  $S'$  over  $\mathcal{O}_{\check{E}}$ , a  $p$ -divisible group  $X$  over  $S'$ , and a quasi-isogeny  $\eta : \mathbf{X}_{\overline{S'}} \rightarrow X_{\overline{S'}}$  for  $\overline{S'} = V(p) \subset S'$ , such that the  $S'$ -valued point  $(X, \eta) \in \check{\mathcal{G}}'(S')$  induces under the period morphism  $\check{\pi}' : \check{\mathcal{G}}'^{\mathrm{an}} \rightarrow \check{\mathcal{F}}_{\rho(b)}^0$  an affinoid subdomain  $S'^{\mathrm{an}} \subset \check{\mathcal{F}}_{\rho(b)}^0$  containing  $\mu$ . By changing  $\mu_0$  we may assume that  $S'^{\mathrm{an}}$  also contains  $\mu_0$ . Let  $S \subset S'$  be the Zariski closed formal subscheme with  $S^{\mathrm{an}} = \check{\mathcal{F}}_b^0 \times_{\check{\mathcal{F}}_{\rho(b)}^0} S'^{\mathrm{an}}$ . We denote by  $V_p X$  the  $\pi_1^{\acute{e}t}(S^{\mathrm{an}}, \bar{\mu}_0)$ -module  $V_p X_{\bar{\mu}_0}$  at some fixed geometric base point  $\bar{\mu}_0$  of  $S^{\mathrm{an}}$  above  $\mu_0$ . By Wintenberger [Win97, Corollary to Proposition 4.5.3] for  $(G', \rho(b))$  there is an isomorphism  $\delta : V_p X \xrightarrow{\sim} \tilde{V}$ .

Consider the polarized multichain of type  $(\mathcal{L})$  in  $\tilde{V}$  constructed in part 1 above from an  $\mathcal{L}$ -set of  $p$ -divisible groups with  $\mathcal{O}_B$ -action above  $x_0 \in \check{\mathcal{G}}(\mathcal{O}_L)$ , which we now call  $\{\bar{X}_\Lambda\}_{\Lambda \in \mathcal{L}}$ . We claim that there is a projective morphism  $T \rightarrow S$  of admissible formal  $\mathcal{O}_{\check{E}}$ -schemes with  $T^{\mathrm{an}} \rightarrow S^{\mathrm{an}}$  a finite étale covering, and for each  $\Lambda \in \mathcal{L}$  a  $p$ -divisible group  $X_\Lambda$  over  $T$  together with a quasi-isogeny  $\eta_{X, \Lambda} : X_\Lambda \rightarrow X$  with  $\delta_\Lambda(T_p X_\Lambda) = \tilde{\Lambda} \subset \tilde{V}$  for  $\delta_\Lambda := \delta \circ V_p(\eta_{X, \Lambda}) : V_p X_\Lambda \xrightarrow{\sim} \tilde{V}$ .

Indeed, by the periodicity  $\tilde{p}\tilde{\Lambda} = p \cdot \tilde{\Lambda}$  which allows to set  $\eta_{p^n \Lambda, \Lambda} := p^{-n} : X_\Lambda \rightarrow X_\Lambda =: X_{p^n \Lambda}$ , we only need to check this for finitely many  $\Lambda_\nu \in \mathcal{L}$  with  $\tilde{\Lambda}_\nu \supset \delta(T_p X)$ . Let  $T^{\mathrm{an}} \rightarrow S^{\mathrm{an}}$  be the finite étale covering corresponding to the  $\pi_1^{\acute{e}t}(S^{\mathrm{an}}, \bar{\mu}_0)$ -set  $\prod_\nu \delta^{-1} \tilde{\Lambda}_\nu / T_p X$ . That is,  $\bar{\mu}_0$  lifts to a point of  $T^{\mathrm{an}}$  and  $\pi_1^{\acute{e}t}(T^{\mathrm{an}}, \bar{\mu}_0)$  acts trivially on  $\prod_\nu \delta^{-1} \tilde{\Lambda}_\nu / T_p X$ . This implies the existence of a  $p$ -divisible group  $X_{\Lambda_\nu}$  over  $T^{\mathrm{an}}$  and a quasi-isogeny  $\eta_{\Lambda_\nu, X} : X \rightarrow X_{\Lambda_\nu}$  with  $\ker \eta_{\Lambda_\nu, X} = (\delta^{-1} \tilde{\Lambda}_\nu / T_p X)_{T^{\mathrm{an}}}$ . The family of finite flat subgroup schemes  $\{\ker \eta_{\Lambda_\nu, X} \subset X\}_\nu$  corresponds to a morphism  $f$  from  $T^{\mathrm{an}}$  to a moduli space of families of finite flat subgroup schemes of  $X$ . This moduli space is projective over  $S$ . Let  $T$  be the scheme theoretic closure of the graph of this morphism  $f$ . Then  $T \rightarrow S$  is projective,  $\ker \eta_{\Lambda_\nu, X}$  extends to a finite flat subgroup scheme of  $X$  over  $T$ , and we set  $\eta_{\Lambda_\nu, X} : X \rightarrow X_{\Lambda_\nu} := X / \ker \eta_{\Lambda_\nu, X}$  and  $\eta_{X, \Lambda_\nu} := \eta_{\Lambda_\nu, X}^{-1}$ . This proves the claim.

We may replace  $S$  by  $T$  and  $(X, \eta)$  by  $(X_\Lambda, \eta_{X, \Lambda}^{-1} \circ \eta)$  for any  $\Lambda$ . The construction yields that the fiber over  $\bar{\mu}_0$  of  $X_\Lambda$  is isomorphic to  $\bar{X}_\Lambda$ . This implies that  $V_p X_\Lambda$  is a  $B$ -module and  $\delta_\Lambda$  is a  $B$ -isomorphism.

3. The quasi-isogeny  $\eta_\Lambda := \eta_{X, \Lambda}^{-1} \circ \eta : \mathbf{X}_{\overline{S}} \rightarrow X_{\Lambda, \overline{S}}$  induces an action  $i_\Lambda(a) = \eta_\Lambda \circ i_{\mathbf{X}}(a) \circ \eta_\Lambda^{-1}$  of  $a \in B$  as quasi-isogenies on  $X_{\Lambda, \overline{S}}$ . For each  $a \in B$  there is an integer  $n$  such that  $i_\Lambda(p^n a)$

is even an isogeny. We denote by  $\mathbb{D}(X_\Lambda)_S$  the Lie algebra of the universal vector extension of  $X_\Lambda$  over  $S$ . It only depends on the reduction  $X_{\Lambda, \bar{S}}$ . We set  $\mathbb{D}(X_\Lambda)_{S^{\text{an}}} = \mathbb{D}(X_\Lambda)_S \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{\text{an}}}$  and denote by  $\mathbb{D}(\eta_\Lambda)_{S^{\text{an}}} : V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}} = \mathbb{D}(\mathbf{X})_{K_0} \otimes_{K_0} \mathcal{O}_{S^{\text{an}}} \xrightarrow{\sim} \mathbb{D}(X_\Lambda)_{S^{\text{an}}}$  the isomorphism induced from  $\eta_\Lambda$  by the crystalline nature of  $\mathbb{D}(X_\Lambda)_S$ . By definition of the period morphism  $\tilde{\pi}'$

$$Fil^1 X_\Lambda = \mathbb{D}(X_\Lambda)_S \cap (Fil^1 X_\Lambda) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{\text{an}}} = \mathbb{D}(X_\Lambda)_S \cap \mathbb{D}(\eta_\Lambda)_{S^{\text{an}}} (Fil^1(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}}))$$

for the universal filtration  $Fil^1(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})$ . Indeed, the right equation holds by definition and for the left equation the inclusion “ $\subset$ ” is clear. On the other hand, if  $m$  belongs to the right hand side then  $p^n m \in Fil^1 X_\Lambda$  for  $n \gg 0$ . Thus the image  $\bar{m}$  of  $m$  in  $\text{Lie } X_\Lambda = \mathbb{D}(X_\Lambda)_S / Fil^1 X_\Lambda$  satisfies  $p^n \bar{m} = 0$ . But  $\text{Lie } X_\Lambda$ , as a locally free  $\mathcal{O}_S$ -module, is without  $p$ -torsion, and so  $m \in Fil^1 X_\Lambda$ .

Since  $Fil^1(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})$  is  $B$ -invariant,  $i_\Lambda(p^n a)$  lifts to an isogeny of  $X_\Lambda$  over  $S$ . Hence  $i_\Lambda(a) := i_\Lambda(p^n a) \otimes p^{-n} \in \text{End}(X_\Lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ . By construction the  $B$ -module structure on  $V_p X_\Lambda$  induced from  $V_p(i_\Lambda(a))$  coincides with the structure considered at the end of part 2 above. For  $a \in \mathcal{O}_B$  then  $a \cdot \tilde{\Lambda} \subset \tilde{\Lambda}$  implies  $V_p(i_\Lambda(a))(T_p X_\Lambda) \subset T_p X_\Lambda$  and therefore  $i_\Lambda(a) \in \text{End}(X_\Lambda)$ . In order that  $(X_\Lambda, i_\Lambda, \eta_\Lambda)_{\Lambda \in \mathcal{L}}$  is an  $S$ -valued point of  $\mathcal{G}$  we have to verify conditions (i)–(v) of [RZ96, Definition 3.21].

By construction  $\mathbb{D}(\eta_\Lambda)_{S^{\text{an}}}$  induces an isomorphism of exact sequences of  $B$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & (Fil^1 X_\Lambda) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{\text{an}}} & \longrightarrow & \mathbb{D}(X_\Lambda)_{S^{\text{an}}} & \longrightarrow & \text{Lie } X_\Lambda \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{\text{an}}} \longrightarrow 0 \\ & & \uparrow \sim & & \uparrow \sim \mathbb{D}(\eta_\Lambda)_{S^{\text{an}}} & & \uparrow \sim \\ 0 & \longrightarrow & Fil^1(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}}) & \longrightarrow & \mathbb{D}(\mathbf{X})_{K_0} \otimes_{K_0} \mathcal{O}_{S^{\text{an}}} & \longrightarrow & gr^0(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}}) \longrightarrow 0 \\ & & \parallel & & \parallel & & \uparrow \sim \\ 0 & \longrightarrow & (V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})^{\mu_{S^{\text{an}}}^{\text{univ}}(p)=p} & \longrightarrow & V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}} & \longrightarrow & (V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})^{\mu_{S^{\text{an}}}^{\text{univ}}(p)=1} \longrightarrow 0, \end{array}$$

where  $\mu_{S^{\text{an}}}^{\text{univ}}$  is the universal cocharacter on  $S^{\text{an}}$ . Hence  $\det_S(a; \text{Lie } X_\Lambda) = \det_{S^{\text{an}}}(a; (V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})^{\mu_{S^{\text{an}}}^{\text{univ}}(p)=1})$  for every  $a \in \mathcal{O}_B$ . This implies condition (iv). Note that (iv) further implies (i) by [RZ96, 3.23(c)].

Next, for any  $\Lambda \subset \Lambda'$  in  $\mathcal{L}$  our condition  $\tilde{\Lambda} \subset \tilde{\Lambda}'$  implies that  $V_p(\eta_{\Lambda', \Lambda})(T_p X_\Lambda) \subset T_p X_{\Lambda'}$  and so  $\eta_{\Lambda', \Lambda} : X_\Lambda \rightarrow X_{\Lambda'}$  is an isogeny. Under the decompositions  $\mathcal{O}_B \cong \prod_{i=1}^r M_{n_i}(\mathcal{O}_{D_i})$ ,  $\Lambda \cong \bigoplus_i \Lambda_i$  and  $X_\Lambda \cong \prod_{i=1}^r X_{\Lambda_i}$  it induces an isogeny  $X_{\Lambda_i} \rightarrow X_{\Lambda'_i}$  of height equal to  $\log_p [T_p X_{\Lambda'_i} : V_p(\eta_{\Lambda', \Lambda})(T_p X_{\Lambda_i})] = \log_p [\tilde{\Lambda}'_i : \tilde{\Lambda}_i] = \log_p [\Lambda'_i : \Lambda_i]$ . Condition (ii) follows from this by [RZ96, 3.23(d)].

If  $a \in B^\times$  satisfies  $a^{-1} \mathcal{O}_B a = \mathcal{O}_B$  then the quasi-isogeny  $\theta_{a, \Lambda} = \eta_{a\Lambda, \Lambda} \circ i_\Lambda(a) : X_\Lambda \rightarrow X_{a\Lambda}$  satisfies

$$V_p(\theta_{a, \Lambda})(T_p X_\Lambda) = V_p(\theta_{a, \Lambda}) \circ \delta_\Lambda^{-1}(\tilde{\Lambda}) = V_p(\eta_{a\Lambda, \Lambda}) \circ \delta_\Lambda^{-1}(a \cdot \tilde{\Lambda}) = \delta_{a\Lambda}^{-1}(a \tilde{\Lambda}) = T_p X_{a\Lambda}.$$

Hence  $\theta_{a, \Lambda} : X_\Lambda \rightarrow X_{a\Lambda}$  is an isomorphism and this proves condition (iii).

In case PEL the quasi-isogeny  $\lambda : \mathbf{X} \rightarrow \mathbf{X}^\vee$  induces a quasi-isogeny

$$\bar{p}_{\Lambda^\vee} := (\eta_{\Lambda^\vee}^\vee)^{-1} \circ \lambda \circ \eta_{\Lambda^\vee}^{-1} : X_{\Lambda^\vee, \bar{S}} \longrightarrow (X_{\Lambda, \bar{S}})^\vee.$$

By the crystalline nature of  $\mathbb{D}(\cdot)_{S^{\text{an}}}$  we obtain an isomorphism  $\mathbb{D}(\bar{p}_{\Lambda^\vee})_{S^{\text{an}}} : \mathbb{D}(X_{\Lambda^\vee})_{S^{\text{an}}} \xrightarrow{\sim} \mathbb{D}(X_\Lambda^\vee)_{S^{\text{an}}} \cong \mathbb{D}(X_\Lambda)_{S^{\text{an}}}^\vee$  where the last isomorphism is induced by the pairing  $\psi_\Lambda$ . Since the

alternating form  $(\cdot, \cdot)$  on  $V \otimes_{\mathbb{Q}_p} K_0 = \mathbb{D}(\mathbf{X})_{K_0}$  coincides with

$$\mathbb{D}(\mathbf{X})_{K_0} \otimes_{K_0} \mathbb{D}(\mathbf{X})_{K_0} \xrightarrow{\mathbb{D}(\lambda) \otimes \text{id}} \mathbb{D}(\mathbf{X}^\vee)_{K_0} \otimes_{K_0} \mathbb{D}(\mathbf{X})_{K_0} \xrightarrow{\mathbb{D}(\psi_{\mathbf{X}})_{K_0}} \mathbb{D}(\mathbf{G}_m)_{K_0} = K_0$$

up to multiplication with a unit in  $\mathbb{Q}_p^\times$  and since  $\text{Fil}^1(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})$  is totally isotropic for  $(\cdot, \cdot)$  we see that

$$\begin{aligned} \mathbb{D}(\bar{p}_{\Lambda^\vee})_{S^{\text{an}}}(\text{Fil}^1 X_{\Lambda^\vee}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{\text{an}}} &= \mathbb{D}(\eta_\Lambda^{-1})^\vee \mathbb{D}(\lambda)(\text{Fil}^1(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})) \\ &\subset \mathbb{D}(\eta_\Lambda^{-1})^\vee(\{h \in V^\vee \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}} : h(\text{Fil}^1(V \otimes_{\mathbb{Q}_p} \mathcal{O}_{S^{\text{an}}})) = 0\}) \\ &= \{h' \in \mathbb{D}(X_\Lambda)_{S^{\text{an}}}^\vee : h'(\text{Fil}^1 X_\Lambda) = 0\} \\ &= (\text{Fil}^1 X_\Lambda^\vee) \otimes_S \mathcal{O}_{S^{\text{an}}}. \end{aligned}$$

This shows that  $\bar{p}_{\Lambda^\vee}$  lifts to a quasi-isogeny  $p_{\Lambda^\vee} : X_{\Lambda^\vee} \rightarrow X_\Lambda^\vee$  over  $S$ . The quasi-isogeny  $p_{\Lambda^\vee} \circ \eta_{\Lambda^\vee, \Lambda} : X_\Lambda \rightarrow X_\Lambda^\vee$  induces an alternating form on  $\tilde{V} \cong V_p X_\Lambda$  as in (7.3) which equals  $\tilde{\ell} \cdot (\cdot, \cdot)^\sim$  for a constant  $\tilde{\ell} \in \mathbb{Q}_p^\times$  by part 1 above. This  $\tilde{\ell}$  is also the constant from Definition 7.3( $\tilde{v}$ ) of the multichain  $\{\tilde{\Lambda}\}_{\Lambda \in \mathcal{L}}$ . Then

$$\begin{aligned} V_p(p_{\Lambda^\vee})(T_p X_{\Lambda^\vee}) &= V_p(\frac{1}{\tilde{\ell}} p_{\Lambda^\vee} \circ \eta_{\Lambda^\vee, \Lambda}) \delta_\Lambda^{-1}(\tilde{\ell} \cdot \tilde{\Lambda}^\vee) \\ &= \frac{1}{\tilde{\ell}} V_p(p_{\Lambda^\vee} \circ \eta_{\Lambda^\vee, \Lambda}) \delta_\Lambda^{-1}(\tilde{\Lambda}^\vee) \\ &= \frac{1}{\tilde{\ell}} V_p(p_{\Lambda^\vee} \circ \eta_{\Lambda^\vee, \Lambda}) \delta_\Lambda^{-1}(\{\tilde{v} \in \tilde{V} : (\tilde{v}, \tilde{w})^\sim \in \mathbb{Z}_p \text{ for all } \tilde{w} \in \tilde{\Lambda}\}) \\ &= T_p X_\Lambda^\vee. \end{aligned}$$

Therefore  $p_{\Lambda^\vee} : X_{\Lambda^\vee} \rightarrow X_\Lambda^\vee$  is an isomorphism and this establishes [RZ96, Definition 3.21(v)]. Hence  $(X_\Lambda, i_\Lambda, \eta_\Lambda)$  is a point in  $\check{\mathcal{G}}(S)$  such that the induced morphism  $S^{\text{an}} \rightarrow \check{\mathcal{G}}^{\text{an}} \xrightarrow{\check{\pi}} \check{\mathcal{F}}_b^0$  coincides with the morphism constructed in part 2 above. This shows that  $\mu \in \check{\mathcal{F}}_b^0$  lies in the image of  $\check{\pi}$  and proves (a).

(b) As in the proof of Theorem 6.3(b) the rational Tate module  $V_p X_\Lambda$  descends to a local system  $\mathcal{V}$  of  $\mathbb{Q}_p$ -vector spaces on  $\check{\mathcal{F}}_b^0$ . Consider a connected component  $Y$  of  $\check{\mathcal{F}}_b^0$  and a geometric base point  $\bar{\mu} \in Y$  which lies over an  $L$ -valued point  $\mu \in Y(L)$  for a finite extension  $L/\check{E}$ . By Proposition 1.2 the local system  $\mathcal{V}$  on  $Y$  corresponds to a representation  $\rho' : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \text{GL}(\mathcal{V}_{\bar{\mu}}) \cong \text{GL}(\tilde{V})$  where we identify  $V_p X_\Lambda \cong \tilde{V}$  as in part 1 of (a) above. There is even an isomorphism of  $\text{Gal}(L^{\text{alg}}/L)$ -representations

$$\mathcal{V}_\mu = V_p X_{\Lambda, \mathcal{O}_L} \cong V_{\text{cris}}(\mathbb{D}(X_{\Lambda, \mathcal{O}_L}), \text{Fil}^1 X_{\Lambda, \mathcal{O}_L}) \cong V_{\text{cris}}(\underline{D}_{b, \mu}(V)). \quad (7.4)$$

In order to prove that  $\mathcal{V}$  induces a tensor functor and (7.4) is a tensor isomorphism it suffices by Corollary 1.5 to show that  $\rho'$  factors through  $\tilde{G}(\mathbb{Q}_p) \subset \text{GL}(\tilde{V})$ , which in turn follows from the explicit description of  $\tilde{G}$  given in (7.2). Indeed, since the universal  $X_\Lambda$  over  $\check{\mathcal{G}}$  carries an action of  $\mathcal{O}_B$  the image of  $\pi_1^{\text{ét}}(Y, \bar{\mu})$  in  $\text{GL}(\tilde{V})$  commutes with  $B$  and this already proves the claim in case EL. In case PEL diagram (7.3) shows that the action of  $\gamma \in \pi_1^{\text{ét}}(Y, \bar{\mu})$  on  $\tilde{V}$  satisfies  $(\gamma \cdot \tilde{v}, \gamma \cdot \tilde{w})^\sim = \chi_{\text{cyc}}(\gamma) \cdot (\tilde{v}, \tilde{w})^\sim$  for all  $\tilde{v}, \tilde{w} \in \tilde{V}$ , where  $\chi_{\text{cyc}} : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \text{Gal}(\check{E}^{\text{alg}}/\check{E}) \xrightarrow{\chi_{\text{cyc}}} \mathbb{Z}_p^\times$  is the cyclotomic character of the base field  $\check{E}$ .

In particular,  $\rho'$  factors through a morphism  $\rho' : \pi_1^{\text{ét}}(Y, \bar{\mu}) \rightarrow \tilde{G}(\mathbb{Q}_p)$  with  $\tilde{c} \circ \rho' = \chi_{\text{cyc}}$ , and by Corollary 1.5 this induces the desired tensor functor  $\underline{\mathcal{V}} : \underline{\text{Rep}}_{\mathbb{Q}_p} G \rightarrow \mathbb{Q}_p\text{-}\underline{\text{Loc}}_Y$  with  $\underline{\mathcal{V}}(V) = \mathcal{V}$  satisfying (1.9). Moreover, (7.4) is compatible with the action of  $B$  on both sides and, in the PEL case, the alternating pairings  $V_p \psi_\Lambda$  and  $V_{\text{cris}}(\mathbb{D}(\psi_{\mathbf{X}})_{K_0})$ ; see (7.3). Hence (7.4) induces an isomorphism of tensor functors on  $\underline{\text{Rep}}_{\mathbb{Q}_p} G$  and also (1.8) of Conjecture 1.3 holds at the fixed  $\mu$ . But the construction is independent of  $\mu$ . So it holds on all connected components  $Y$  of  $\check{\mathcal{F}}_b^0$  and this establishes (b) and Remark 7.5.

(c) The argument of Theorem 6.3(c) applies literally to prove the first part of (c). For the statement about  $\check{\mathcal{G}}^{\text{an}}$  let  $\check{K} \subset \tilde{G}(\mathbb{Q}_p)$  be the stabilizer of the (polarized) multichain  $(\tilde{\Lambda})_{\Lambda \in \mathcal{L}}$  of type  $(\mathcal{L})$  in  $\tilde{V}$  constructed in part 1 above. The construction of parts 2 and 3 yields an isomorphism  $\check{\mathcal{G}}^{\text{an}} \xrightarrow{\sim} \check{\mathcal{G}}_{\check{K}}^{\text{an}} \cong \check{\mathcal{E}}_{\check{K}}$  where  $\check{\mathcal{E}}_{\check{K}}$  is the space of (polarized) multichains of type  $(\mathcal{L})$  in  $\mathcal{V}$ .  $\square$

## 8 When does weakly admissible imply admissible ?

In this section we approach the question for which groups  $G$  and elements  $b \in G(K_0)$  as in (5.1) “weakly admissible implies admissible”, that is  $\check{\mathcal{F}}_b^{\text{wa}} = \check{\mathcal{F}}_b^0$ . Since we do not yet have a good group theoretic understanding of the phenomenon we only treat the case  $G = \text{GL}_n$  comprehensively. We begin with the following observations

**Lemma 8.1.** *Let  $G = G_1 \times G_2$  be a product of two reductive groups over  $\mathbb{Q}_p$  and  $b = (b_1, b_2)$  such that each  $b_i \in G_i(K_0)$  satisfies (5.1) for the same integer  $n$ . If  $\check{\mathcal{F}}_{G_1, b_1}^{\text{wa}} \neq \check{\mathcal{F}}_{G_1, b_1}^0$  and  $\check{\mathcal{F}}_{G_2, b_2}^{\text{wa}} \neq \emptyset$  then  $\check{\mathcal{F}}_{G, b}^{\text{wa}} \neq \check{\mathcal{F}}_{G, b}^0$ .*

*Proof.* Let  $\mu_1 \in \check{\mathcal{F}}_{G_1, b_1}^{\text{wa}} \setminus \check{\mathcal{F}}_{G_1, b_1}^0$  be an  $L_1$ -valued point and  $\mu_2 \in \check{\mathcal{F}}_{G_2, b_2}^{\text{wa}}(L_2)$  for complete field extensions  $L_1/\check{E}_1$  and  $L_2/\check{E}_2$  with  $[L_2 : \check{E}_2] < \infty$ . Let  $L$  be some compositum of  $L_1$  and  $L_2$  in some algebraic closure of  $L_1$  and  $\mu = (\mu_1, \mu_2) : \mathbb{G}_m \rightarrow G$ . Let  $\rho_i : G_i \hookrightarrow \text{GL}(V_i)$  be representations satisfying (5.1) for  $n$ . Then  $\rho : G \hookrightarrow \text{GL}(V_1 \oplus V_2)$  likewise satisfies (5.1) for  $n$ . Consider the filtered isocrystals  $\underline{D}_i = \underline{D}_{b_i, \mu_i}(V_i)$  and  $\underline{D} = \underline{D}_1 \oplus \underline{D}_2 = \underline{D}_{b, \mu}(V_1 \oplus V_2)$  over  $L$  which all three are weakly admissible. Then  $\mathbf{M}(\underline{D}) = \mathbf{M}(\underline{D}_1) \oplus \mathbf{M}(\underline{D}_2) \not\cong \mathbf{M}(0, 1)^{\dim V_1 + \dim V_2}$  because  $\mathbf{M}(\underline{D}_1) \not\cong \mathbf{M}(0, 1)^{\dim V_1}$ . This shows that  $\mu \in \check{\mathcal{F}}_{G, b}^{\text{wa}} \setminus \check{\mathcal{F}}_{G, b}^0$  as desired.  $\square$

The next lemma shows that the question is invariant under dualization.

**Lemma 8.2.** *Let  $G_i = \text{GL}_n$  and  $b_i \in \text{GL}_n(K_0)$  for  $i = 1, 2$  such that the isocrystals  $\underline{D}_i = (K_0^n, b_i \cdot \varphi)$  satisfy (5.1). Assume that*

$$(a) \underline{D}_2 \cong \underline{D}_1 \otimes \mathbf{1}(r) \text{ or}$$

$$(b) \underline{D}_2 \cong \text{Hom}(\underline{D}_1, \mathbf{1}(r))$$

*holds for some  $r \in \mathbb{Z}$  (see Lemma 4.3). Then  $\check{\mathcal{F}}_{G_1, b_1}^{\text{wa}} = \check{\mathcal{F}}_{G_1, b_1}^0$  if and only if  $\check{\mathcal{F}}_{G_2, b_2}^{\text{wa}} = \check{\mathcal{F}}_{G_2, b_2}^0$ .*

*Proof.* By symmetry it suffices to prove one direction. So assume that  $\check{\mathcal{F}}_{G_2, b_2}^{\text{wa}} = \check{\mathcal{F}}_{G_2, b_2}^0$  and let  $\mu_1 \in \check{\mathcal{F}}_{G_1, b_1}^{\text{wa}}$ . Since the filtered isocrystal  $\underline{D}_1 = (K_0^n, b_1 \cdot \varphi, \text{Fil}_{\mu_1}^\bullet)$  is weakly admissible, also the filtered isocrystal  $\underline{D}_2 := \underline{D}_1 \otimes \mathbf{1}(r)$  in case (a), respectively  $\underline{D}_2 := \text{Hom}(\underline{D}_1, \mathbf{1}(r))$  in case (b) is weakly admissible and corresponds to a point  $\mu_2 \in \check{\mathcal{F}}_{G_2, b_2}^{\text{wa}}$ . Then our assumption implies  $\mathbf{M}(\underline{D}_2) \cong \mathbf{M}(0, 1)^n$ . Lemma 4.3 and Proposition 4.4 yield in case

- (a)  $\mathbf{M}(\underline{D}_1) \cong \mathbf{M}(\underline{D}_2) \cong \mathbf{M}(0, 1)^n$ , respectively
- (b)  $\mathbf{M}(\underline{D}_1) \cong \mathbf{M}(\underline{D}_2^\vee \otimes \mathbb{1}(r)) \cong \mathbf{M}(\underline{D}_2^\vee) \cong (\mathbf{M}(0, 1)^n)^\vee \cong \mathbf{M}(0, 1)^n$ ,

and therefore  $\mu_1 \in \check{\mathcal{F}}_{G_1, b_1}^0$  as desired.  $\square$

Next we state the main theorem of this section.

**Theorem 8.3.** *Let  $G = \mathrm{GL}_n$ , let  $b \in \mathrm{GL}_n(K_0)$  such that the isocrystal  $(K_0^n, b \cdot \varphi)$  has Newton slopes in the interval  $[0, 1]$  and let  $\{\mu_0\}$  be the conjugacy class for which the standard representation  $\mathbb{Q}_p^n$  of  $G$  satisfies (5.1). Then  $\check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}_b^0$  if and only if the Newton slopes of  $b$  are*

- (a)  $(1^{(h_1)}, \frac{1}{h}^{(h)}, 0^{(h_0)})$  for  $h_1, h, h_0 \in \mathbb{N}_0$ , or
- (b)  $(1^{(h_1)}, \frac{h-1}{h}^{(h)}, 0^{(h_0)})$  for  $h_1, h, h_0 \in \mathbb{N}_0$ , or
- (c)  $(1^{(h_1)}, \frac{1}{2}^{(4)}, 0^{(h_0)})$  for  $h_1, h_0 \in \mathbb{N}_0$ .

In particular  $\check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}_b^0$  for  $n \leq 4$ . (Here  $s^{(m)}$  means that the slope  $s \in \mathbb{Q}$  occurs with multiplicity  $m \in \mathbb{N}_0$ . If  $m = 0$  this slope does not occur. The sum of all multiplicities is  $n$ .)

Before proving the theorem we derive the following

**Corollary 8.4.** *Let  $(G, b, \{\mu\})$  be arbitrary such that there is a faithful representation  $\rho : G \hookrightarrow \mathrm{GL}(V)$  as in (5.1) with  $\dim_{\mathbb{Q}_p} V \leq 4$ . Then  $\check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}_b^0$ .*

*Proof.* This follows from the theorem and the fact that  $\check{\mathcal{F}}_{G, b}^0 = \check{\mathcal{F}}_{G, b}^{wa} \cap \check{\mathcal{F}}_{\mathrm{GL}(V), \rho(b)}^0$ , see (5.2) and Proposition 5.2.  $\square$

In particular  $\check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}_b^0$  in the PEL situation of Section 7 with  $G = \mathrm{GSp}_4$ .

*Proof of Theorem 8.3.* We first prove that the conditions (a), (b), and (c) are sufficient. Let  $\mu \in \check{\mathcal{F}}^{\mathrm{an}} \setminus \check{\mathcal{F}}_b^0$ . We must show that  $\mu \notin \check{\mathcal{F}}_b^{wa}$ . We set  $\mathbf{M}_\mu = \mathbf{M}(\underline{D}_{b, \mu}(\mathbb{Q}_p^n))$  and  $\mathbf{D} = \mathbf{D}(\underline{D}_{b, \mu}(\mathbb{Q}_p^n))$  (Definition 4.2), and consider the decompositions of  $\mathbf{D}$  and  $\mathbf{M}_\mu$  from Theorem 3.3. We write  $\mathbf{M}_\mu = \mathbf{M}_+ \oplus \mathbf{M}_0 \oplus \mathbf{M}_-$  such that the weights of all components of  $\mathbf{M}_+$  (respectively  $\mathbf{M}_0$ , respectively  $\mathbf{M}_-$ ) are positive (respectively zero, respectively negative). Note that  $\mathbf{M}_- \neq (0)$  since  $\mathbf{M}_\mu \not\cong \mathbf{M}(0, 1)^n$  and  $\deg \mathbf{M}_\mu = 0$  by Theorem 4.5 and Lemma 5.1. Likewise we write  $\mathbf{D} = \mathbf{M}(1, 1)^{\oplus h_1} \oplus \mathbf{D}_+ \oplus \mathbf{M}(0, 1)^{\oplus h_0}$  such that the weights of all components of  $\mathbf{D}_+$  lie strictly between 0 and 1. Then  $t^{-1}\mathbf{D} = \mathbf{M}(0, 1)^{\oplus h_1} \oplus t^{-1}\mathbf{D}_+ \oplus \mathbf{M}(-1, 1)^{\oplus h_0}$ . We consider the inclusions  $\mathbf{D} \hookrightarrow \mathbf{M}_\mu \hookrightarrow t^{-1}\mathbf{D}$  whose composition is the natural inclusion  $\mathbf{D} \subset t^{-1}\mathbf{D}$ , and the induced maps  $\mathbf{M}(0, 1)^{\oplus h_0} \rightarrow \mathbf{M}_0 \rightarrow \mathbf{M}(0, 1)^{\oplus h_1}$  whose composition is the zero map. Since  $\mathrm{End}_\varphi(\mathbf{M}(0, 1)) = \mathbb{Q}_p$  (use [Ked05, Proposition 3.3.4]) we can decompose  $\mathbf{M}_\mu = \mathbf{M}_+ \oplus \mathbf{M}_0^1 \oplus \mathbf{M}_0^2 \oplus \mathbf{M}_0^3 \oplus \mathbf{M}_-$ , and  $\mathbf{D} = \mathbf{M}(1, 1)^{\oplus h_1} \oplus \mathbf{M}(1, 1)^{\oplus h'_1} \oplus \mathbf{D}_+ \oplus \mathbf{M}(0, 1)^{\oplus h'_0} \oplus \mathbf{M}(0, 1)^{\oplus \tilde{h}_0}$  such that with respect to appropriate bases of all these  $\varphi$ -modules (and the induced basis for  $t^{-1}\mathbf{D}$ ) the inclusions are given by block matrices

$\mathbf{M}_\mu \hookrightarrow t^{-1}\mathbf{D}$	$\mathbf{M}_+$	$\mathbf{M}_0^1$	$\mathbf{M}_0^2$	$\mathbf{M}_0^3$	$\mathbf{M}_-$
$\mathbf{M}(0, 1)^{\oplus \tilde{h}_1}$	$A_1$	0	0	0	0
$\mathbf{M}(0, 1)^{\oplus h'_1}$	$A_2$	Id	0	0	0
$t^{-1}\mathbf{D}_+$	$D$	$E_1$	$E_2$	$E_3$	$F$
$\mathbf{M}(-1, 1)^{\oplus h'_0}$	$G_1$	$H_{11}$	$H_{12}$	$H_{13}$	$I_1$
$\mathbf{M}(-1, 1)^{\oplus \tilde{h}_0}$	$G_2$	$H_{21}$	$H_{22}$	$H_{23}$	$I_2$

and

$$\begin{array}{c|ccccc}
\mathbf{D} \hookrightarrow \mathbf{M}_\mu & \mathbf{M}(1,1)^{\oplus \tilde{h}_1} & \mathbf{M}(1,1)^{\oplus h'_1} & \mathbf{D}_+ & \mathbf{M}(0,1)^{\oplus h'_0} & \mathbf{M}(0,1)^{\oplus \tilde{h}_0} \\
\hline
\mathbf{M}_+ & a_1 & a_2 & c & 0 & 0 \\
\mathbf{M}_0^1 & d_{11} & d_{12} & e_1 & 0 & 0 \\
\mathbf{M}_0^2 & d_{21} & d_{22} & e_2 & 0 & 0 \\
\mathbf{M}_0^3 & d_{31} & d_{32} & e_3 & \text{Id} & 0 \\
\mathbf{M}_- & f_1 & f_2 & g & i_1 & i_2
\end{array}$$

We put  $r_+ := \text{rk } \mathbf{M}_+$ ,  $r_0^i := \text{rk } \mathbf{M}_0^i$ ,  $r_- := \text{rk } \mathbf{M}_-$ , and  $h := \text{rk } \mathbf{D}_+$ . We must have

$$r_0^1 = h'_1, \quad r_0^3 = h'_0, \quad \tilde{h}_0 \leq r_-, \quad \text{and} \quad r_0^2 + r_0^3 + r_- \leq h + h_0 \quad (8.1)$$

since the maps  $\mathbf{M}(0,1)^{\oplus \tilde{h}_0} \hookrightarrow \mathbf{M}_-$  and  $\mathbf{M}_0^2 \oplus \mathbf{M}_0^3 \oplus \mathbf{M}_- \hookrightarrow t^{-1}\mathbf{D}_+ \oplus \mathbf{M}(-1,1)^{\oplus h_0}$  are injective. The inclusion  $\mathbf{D} \hookrightarrow t^{-1}\mathbf{D}$  is represented by the product of the two big block matrices which hence equals

$$t \text{Id}_n = \begin{pmatrix} A_1 a_1 & A_1 a_2 & A_1 c & 0 & 0 \\ A_2 a_1 + d_{11} & A_2 a_2 + d_{12} & A_2 c + e_1 & 0 & 0 \\ & & & E_3 + F i_1 & F i_2 \\ & & & H_{13} + I_1 i_1 & I_1 i_2 \\ & & & H_{23} + I_2 i_1 & I_2 i_2 \end{pmatrix}$$

This implies  $A_1 c = 0$ ,  $A_2 c + e_1 = 0$ ,  $E_3 + F i_1 = 0$ , and  $F i_2 = 0$ . Considering the entries of these matrices as elements of the fraction field  $\mathcal{Q}$  of the integral domain  $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$  we have  $\dim_{\mathcal{Q}} \ker A_1 \geq \text{rk } c$  and  $\dim_{\mathcal{Q}} \ker F \geq \text{rk } i_2 = \tilde{h}_0$ . Hence

$$\tilde{h}_1 = \text{rk } A_1 \leq r_+ - \text{rk } c \quad \text{and} \quad r_- = \text{rk } F + \dim_{\mathcal{Q}} \ker F \geq \tilde{h}_0 + \text{rk } F. \quad (8.2)$$

Now we distinguish several cases.

Case  $c = 0$ : This implies  $e_1 = 0$  and we obtain inclusions

$$\mathbf{D}' := \mathbf{D}_+ \oplus \mathbf{M}(0,1)^{\oplus h_0} \hookrightarrow \mathbf{M}_0^2 \oplus \mathbf{M}_0^3 \oplus \mathbf{M}_- \hookrightarrow t^{-1}\mathbf{D}_+ \oplus \mathbf{M}(-1,1)^{\oplus h_0}$$

Hence  $\mathbf{D}' \subset \mathbf{D}$  comes from a sub-isocrystal  $\underline{D}' \subset \underline{D}$  with  $\mathbf{M}(\underline{D}') \supset \mathbf{M}_0^2 \oplus \mathbf{M}_0^3 \oplus \mathbf{M}_-$ . This implies  $t_N(\underline{D}') - t_H(\underline{D}') = \deg \mathbf{M}(\underline{D}') \leq \deg(\mathbf{M}_0^2 \oplus \mathbf{M}_0^3 \oplus \mathbf{M}_-) < 0$  by Theorem 4.5 and [Ked05, Lemma 3.4.2]. So  $\mu$  is not weakly admissible,  $\mu \notin \check{\mathcal{F}}_b^{wa}$ .

Case  $F = 0$ : This implies  $E_3 = 0$  and we obtain inclusions

$$\mathbf{D}' := \mathbf{M}(0,1)^{\oplus h_0} \hookrightarrow \mathbf{M}_0^3 \oplus \mathbf{M}_- \hookrightarrow \mathbf{M}(-1,1)^{\oplus h_0}$$

Hence  $\mathbf{D}' \subset \mathbf{D}$  comes from a sub-isocrystal  $\underline{D}' \subset \underline{D}$  with  $\mathbf{M}(\underline{D}') \supset \mathbf{M}_0^3 \oplus \mathbf{M}_-$ . This again implies  $t_N(\underline{D}') - t_H(\underline{D}') < 0$  by Theorem 4.5 and [Ked05, Lemma 3.4.2], and so  $\mu$  is not weakly admissible,  $\mu \notin \check{\mathcal{F}}_b^{wa}$ .

Case  $c \neq 0$  and  $\text{rk } c = h$ : Then (8.2) implies  $\tilde{h}_1 + h \leq r_+$  and

$$\tilde{h}_0 = n - h'_1 - \tilde{h}_1 - h - h'_0 \geq n - r_0^1 - r_+ - r_0^3 = r_0^2 + r_- \geq r_- \geq \tilde{h}_0,$$

implies  $r_- = \tilde{h}_0$ . Since  $t^{-1}I_2 i_2 = \text{Id}_{r_-}$  we have  $i_2 \in \text{GL}_{r_-}(\mathcal{Q})$ . Hence  $F i_2 = 0$  yields  $F = 0$  in which case we just saw that  $\mu \notin \check{\mathcal{F}}_b^{wa}$ .

Case  $F \neq 0$  and  $\text{rk } F = h$  : Then (8.2) implies  $\tilde{h}_0 + h \leq r_-$  and

$$h_0 + h = h'_0 + \tilde{h}_0 + h \leq r_0^3 + r_- \leq r_0^2 + r_0^3 + r_- \leq h_0 + h,$$

implies  $h_0 + h = r_0^2 + r_0^3 + r_-$  and  $\tilde{h}_1 = r_+$ . Since  $t^{-1}A_1a_1 = \text{Id}_{r_+}$  we have  $A_1 \in \text{GL}_{r_+}(\mathcal{Q})$  and hence  $A_1c = 0$  yields  $c = 0$ . This case was treated above.

Case  $c \neq 0$  and  $F \neq 0$  : To treat this case we use the special form  $\mathbf{D}_+$  has under the conditions (a), (b), and (c).

Under condition (a)  $\mathbf{D}_+ \cong \mathbf{M}(1, h)$ . The  $\varphi$ -module which is the image of  $c : \mathbf{D}_+ \rightarrow \mathbf{M}_+$  is isomorphic to  $\bigoplus_i \mathbf{M}(s_i, n_i)$ . Since it is a quotient of  $\mathbf{D}_+$  and contained in  $\mathbf{M}_+$  we have  $\frac{1}{h} \geq \frac{s_i}{n_i} > 0$ , whence  $n_i \geq s_i h \geq h$  for all  $i$ . This means  $\text{rk } c = h$  and we are in a case treated above.

Under condition (b)  $\mathbf{D}_+ \cong \mathbf{M}(h-1, h)$ . The image of  $F : \mathbf{M}_- \rightarrow t^{-1}\mathbf{D}_+$  is isomorphic to  $\bigoplus_i \mathbf{M}(s'_i, n'_i)$ . Since it is a quotient of  $\mathbf{M}_-$  and contained in  $t^{-1}\mathbf{D}_+ \cong \mathbf{M}(-1, h)$  we have  $0 > \frac{s'_i}{n'_i} \geq \frac{-1}{h}$ , whence  $n'_i \geq -s'_i h \geq h$  for all  $i$ . This means  $\text{rk } F = h$  and we are again in a case treated above.

Under condition (c)  $\mathbf{D}_+ \cong \mathbf{M}(1, 2)^{\oplus 2}$ . Again the image of  $c : \mathbf{D}_+ \rightarrow \mathbf{M}_+$  is isomorphic to  $\bigoplus_i \mathbf{M}(s_i, n_i)$  with  $\frac{1}{2} \geq \frac{s_i}{n_i} > 0$ , whence  $n_i \geq 2s_i \geq 2$  and  $\text{rk } c \geq 2$ . Furthermore the image of  $F : \mathbf{M}_- \rightarrow t^{-1}\mathbf{D}_+$  is isomorphic to  $\bigoplus_i \mathbf{M}(s'_i, n'_i)$  with  $0 > \frac{s'_i}{n'_i} \geq \frac{-1}{2}$ , whence  $n'_i \geq -2s'_i \geq 2$  and  $\text{rk } F \geq 2$ . Then

$$n = h_0 + 4 + h_1 \leq h'_0 + \tilde{h}_0 + \text{rk } F + \text{rk } c + \tilde{h}_1 + h'_1 \leq r_0^3 + r_- + r_+ + r_0^1 = n - r_0^2 \leq n$$

yields  $r_0^2 = 0$ ,  $\text{rk } c = 2 = \text{rk } F$  and  $r_- = \tilde{h}_0 + \text{rk } F = \tilde{h}_0 + 2$ . This implies  $n'_i \leq 2$ , whence  $n'_i = 2$ ,  $s'_i = -1$  and  $\text{im}(F) \cong \mathbf{M}(-1, 2)$ . We may decompose  $t^{-1}\mathbf{D}_+ = t^{-1}\mathbf{D}_+^1 \oplus t^{-1}\mathbf{D}_+^2$  with  $\text{im}(F) = t^{-1}\mathbf{D}_+^2$ . With respect to a basis suited to this decomposition we write  $F = \begin{pmatrix} 0 \\ F_2 \end{pmatrix}$ ,  $E_3 = \begin{pmatrix} E_{31} \\ E_{32} \end{pmatrix}$ ,  $c = (c_1|c_2)$  and  $e_1 = (e_{11}|e_{12})$ . Since the zero map

$$\mathbf{M}_- \xrightarrow{\begin{pmatrix} 0 \\ F_2 \end{pmatrix}} t^{-1}\mathbf{D}_+ \xrightarrow{(c_1|c_2)} t^{-1}\mathbf{M}_+$$

equals  $c_2 F_2$  and  $\text{rk } F_2 = 2$  we find  $c_2 = 0$ . Then  $\begin{pmatrix} E_{31} \\ E_{32} \end{pmatrix} + \begin{pmatrix} 0 \\ F_2 \end{pmatrix} i_1 = 0$  implies  $E_{31} = 0$  and  $A_2(c_1|c_2) + (e_{11}|e_{12}) = 0$  implies  $e_{12} = 0$ . We obtain inclusions

$$\mathbf{D}' := \mathbf{D}_+^2 \oplus \mathbf{M}(0, 1)^{\oplus h_0} \hookrightarrow \mathbf{M}_0^3 \oplus \mathbf{M}_- \hookrightarrow t^{-1}\mathbf{D}_+^2 \oplus \mathbf{M}(-1, 1)^{\oplus h_0}$$

The  $\varphi$ -submodule  $\mathbf{D}_+^2$  is inherited from a corresponding sub-isocrystal  $\underline{D}_+^2 := (K_0^n, b \cdot \varphi) \cap \mathbf{D}_+^2$  of the isocrystal  $\underline{D}_+ = (K_0^n, b \cdot \varphi) \cap \mathbf{D}_+$  because all slopes of  $\text{im}(F)$  and  $t^{-1}\mathbf{D}_+$  are equal, and  $\text{End}_\varphi(\mathbf{M}(-1, 2))$  equals the central division algebra over  $\mathbb{Q}_p$  of dimension 4 and Hasse invariant  $\frac{1}{2}$  by Proposition 3.12. Therefore  $\mathbf{D}' \subset \mathbf{D}$  comes from a sub-isocrystal  $\underline{D}' \subset \underline{D}$  with  $\mathbf{M}(\underline{D}') \supset \mathbf{M}_0^3 \oplus \mathbf{M}_-$ . This implies  $t_N(\underline{D}') - t_H(\underline{D}') < 0$  by Theorem 4.5 and [Ked05, Lemma 3.4.2], and so  $\mu$  is not weakly admissible,  $\mu \notin \mathcal{F}_b^{wa}$ .

Altogether we have proved that under conditions (a), (b), and (c)  $\check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}_b^0$ , that is “weakly admissible implies admissible”.

To prove the converse we assume that conditions (a), (b), or (c) are not satisfied. By Lemma 8.1 we only need to find a direct summand of the isocrystal  $(K_0^n, b \cdot \varphi)$  for which “weakly admissible does not imply admissible”. Considering simple sub-isocrystals it hence suffices to treat the cases where  $\mathbf{D} = \mathbf{D}(\underline{D})$  has the following form

1.  $\mathbf{D} \cong \mathbf{M}(c, h)$  and  $t^{-1}\mathbf{D} \cong \mathbf{M}(-d, h)$  for  $c, h \in \mathbb{N}_{>0}$  relatively prime,  $d = h - c$ , with  $2 \leq c, d \leq h - 2$  and  $h \geq 5$ .
2.  $\mathbf{D} \cong \mathbf{M}(1, h_1) \oplus \mathbf{M}(1, h_2)$  with  $2 \leq h_1 \leq h_2$  and  $2 < h_2$ .
3.  $\mathbf{D} \cong \mathbf{M}(h_1 - 1, h_1) \oplus \mathbf{M}(h_2 - 1, h_2)$  with  $2 \leq h_1 \leq h_2$  and  $2 < h_2$ .
4.  $\mathbf{D} \cong \mathbf{M}(1, h_1) \oplus \mathbf{M}(h_2 - 1, h_2)$  with  $3 \leq h_1, h_2$ .
5.  $\mathbf{D} \cong \mathbf{M}(1, 2)^{\oplus 3}$ .

Since the isocrystals in cases 2. and 3. are dual to each other, case 3. follows from case 2. by Lemma 8.2. Also note that a special instance of case 1. was discussed in Example 5.7. However, the beginning of our argument treats cases 1., 2., 4., and 5. simultaneously. We set  $h := \text{rk } \mathbf{D}$ ,  $c := t_N(\underline{D}) = \text{deg } \mathbf{D}$ , and  $d = h - c$ . In case 1. the conditions on  $c, d, h$  imply  $cd > h$  and we define  $e = \lfloor \frac{cd}{h} \rfloor \geq 1$  such that  $\frac{-e}{c} \geq \frac{-d}{h}$ . In the other three cases we set  $e = 1$ . By Proposition 3.12 there exists  $0 \neq A \in \text{Hom}_\varphi(\mathbf{M}(-e, c), t^{-1}\mathbf{D})$ , such that depending on the case

Case 1. the map  $A : \mathbf{M}(-e, c) \rightarrow t^{-1}\mathbf{D}$  is arbitrary but non-zero.

Case 2. the map  $A : \mathbf{M}(-e, c) = \mathbf{M}(-1, 2) \rightarrow t^{-1}\mathbf{D} \cong \mathbf{M}(1 - h_1, h_1) \oplus \mathbf{M}(1 - h_2, h_2)$  induces non-zero maps  $A_1$  and  $A_2$  into both summands of  $t^{-1}\mathbf{D}$ . If  $h_1 = h_2 > 2$  we require in addition that  $A_1, A_2$  are linearly independent over the central division algebra  $\text{End}_\varphi(\mathbf{M}(1 - h_1, h_1))$  of dimension  $h_1^2$  over  $\mathbb{Q}_p$ . This is possible since then  $\dim_{\mathbb{Q}_p} \text{Hom}_\varphi(\mathbf{M}(-1, 2), \mathbf{M}(1 - h_1, h_1)) = \infty$  by Proposition 3.12.

Case 4. the map  $A : \mathbf{M}(-e, c) = \mathbf{M}(-1, h_2) \rightarrow t^{-1}\mathbf{D} \cong \mathbf{M}(1 - h_1, h_1) \oplus \mathbf{M}(-1, h_2)$  induces a non-zero map into the first summand of  $t^{-1}\mathbf{D}$  and is the identity onto the second summand.

Case 5. the map  $A : \mathbf{M}(-e, c) = \mathbf{M}(-1, 3) \rightarrow t^{-1}\mathbf{D} \cong \mathbf{M}(-1, 2)^{\oplus 3}$  induces maps  $A_i$  into the  $i$ -th summand of  $t^{-1}\mathbf{D}$  which are linearly independent over the central division algebra  $\text{End}_\varphi(\mathbf{M}(-1, 2))$  of dimension 4 over  $\mathbb{Q}_p$ . Again this is possible since  $\dim_{\mathbb{Q}_p} \text{Hom}_\varphi(\mathbf{M}(-1, 3), \mathbf{M}(-1, 2)) = \infty$  by Proposition 3.12.

Since both  $\mathbf{M}(-e, c)$  and  $t^{-1}\mathbf{D}$  are represented over  $\tilde{\mathbf{B}}^{[0,1]}(\mathbb{C}_p)$ , we may by Proposition 3.9 choose a  $K_0$ -basis of  $\underline{D}$  and consider  $A$  as an  $(h \times c)$ -matrix with entries in  $\tilde{\mathbf{B}}^{[0,1]}$ . There is a  $\mu$  defined over  $\mathbb{C}_p$  in the conjugacy class  $\{\mu_0\}$ , such that the  $c$ -dimensional space  $\text{Fil}_\mu^1 V_{\mathbb{C}_p}$  contains the columns of the matrix  $\theta_{t^{-1}\mathbf{D}}(A)$ . By Definition 4.2 (compare Example 5.7) the non-zero map  $\mathbf{M}(-e, c) \rightarrow t^{-1}\mathbf{D}$  factors through  $\mathbf{M}_\mu$  for this  $\mu$ , and therefore  $\mathbf{M}_\mu \not\cong \mathbf{M}(0, 1)^{\dim D}$  and  $\mu \notin \check{\mathcal{F}}_b^0$ . It remains to show that  $\mu \in \check{\mathcal{F}}_b^{wa}$ . We distinguish the cases

1. In this case  $\underline{D}$  is simple and hence  $\check{\mathcal{F}}_b^{wa} = \check{\mathcal{F}}_b^{\text{an}}$ . This implies  $\mu \in \check{\mathcal{F}}_b^{wa}$ .

2. Assume that there is a sub-isocrystal  $(0) \neq \underline{D}' \subsetneq \underline{D}$  that contradicts weak admissibility. Then  $t^{-1}\mathbf{D}' := t^{-1}\underline{D}' \otimes_{K_0} \tilde{\mathbf{B}}_{\text{rig}}^\dagger$  is the kernel of a non-zero map  $E : t^{-1}\mathbf{D} \rightarrow \mathbf{M}(1 - h_i, h_i)$  for  $i = 1$  or  $i = 2$ . Also

$$1 = t_N(\underline{D}') < t_H(\underline{D}') = \dim_{\mathbb{C}_p}(D'_{\mathbb{C}_p} \cap \text{Fil}_\mu^1 D_{\mathbb{C}_p}) \leq \dim_{\mathbb{C}_p} \text{Fil}_\mu^1 D_{\mathbb{C}_p} = 2$$

implies  $\text{Fil}_\mu^1 D_{\mathbb{C}_p} \subset D'_{\mathbb{C}_p}$ , whence  $\theta_{\mathbf{M}(1-h_i, h_i)}(EA) = 0$ . Thus  $EA = T \cdot B$  for some map  $B \in \text{Hom}_\varphi(\mathbf{M}(-1, 2), \mathbf{M}(1, h_i))$  by Corollary 3.11. Then Proposition 3.12 shows that

$B = 0$ , whence  $EA = 0$ . However, this is not the case by our assumptions on  $A$ , since for  $h_1 \neq h_2$  the map  $E$  is the projection onto the summand  $\mathbf{M}(1 - h_i, h_i)$  of  $t^{-1}\mathbf{D}$ . Likewise for  $h_1 = h_2$  we have  $E = (E_1|E_2) : t^{-1}\mathbf{D} \rightarrow \mathbf{M}(1 - h_1, h_1)$  with  $E_i \in \text{End}_\varphi(\mathbf{M}(1 - h_1, h_1))$  and  $E_1A_1 + E_2A_2 = 0$  contradicts the linear independence of  $A_1, A_2$  over  $\text{End}_\varphi(\mathbf{M}(1 - h_1, h_1))$ . Therefore  $\mu \in \check{\mathcal{F}}_b^{wa}$ .

4. In this case the matrix  $A$  equals  $\begin{pmatrix} A_1 \\ \text{Id}_{h_2} \end{pmatrix}$  with  $0 \neq A_1 \in \text{Hom}_\varphi(\mathbf{M}(-1, h_2), \mathbf{M}(1 - h_1, h_1))$ , and  $\theta_{t^{-1}\mathbf{D}}(A) = \begin{pmatrix} \theta(A_1) \\ \text{Id}_{h_2} \end{pmatrix} \neq \begin{pmatrix} 0 \\ \text{Id}_{h_2} \end{pmatrix}$  because  $\theta_{\mathbf{M}(1-h_1, h_1)}(A_1) = 0$  would by Corollary 3.11 imply that  $A_1 = T \cdot B$  for  $0 \neq B \in \text{Hom}_\varphi(\mathbf{M}(-1, h_2), \mathbf{M}(1, h_1))$  which is impossible by Proposition 3.12. In particular,  $\theta_{t^{-1}\mathbf{D}}(A)$  has rank  $c$  and  $\text{Fil}_\mu^1 D_{\mathbb{C}_p}$  equals  $\text{im}(\theta_{t^{-1}\mathbf{D}}(A))$ .

There are exactly two sub-isocrystals  $(0) \neq \underline{D}' \subsetneq \underline{D}$ . For the first one  $\mathbf{D}' := \underline{D}' \otimes_{K_0} \tilde{\mathbf{B}}_{\text{rig}}^\dagger = \mathbf{M}(1, h_1)$  and  $t_N(\underline{D}') = 1$ . Also  $D'_{\mathbb{C}_p} \cap \text{Fil}_\mu^1 D_{\mathbb{C}_p} = \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix} \in \text{im} \begin{pmatrix} \theta(A_1) \\ \text{Id}_{h_2} \end{pmatrix} \right\} = (0)$  and  $t_H(\underline{D}') = 0$ . For the second sub-isocrystal  $t^{-1}\mathbf{D}' = \mathbf{M}(-1, h_2)$  and  $t_N(\underline{D}') = h_2 - 1$ . If

$$h_2 - 1 < t_H(\underline{D}') = \dim_{\mathbb{C}_p}(D'_{\mathbb{C}_p} \cap \text{Fil}_\mu^1 D_{\mathbb{C}_p}) \leq \dim_{\mathbb{C}_p} \text{Fil}_\mu^1 D_{\mathbb{C}_p} = h_2$$

then  $\text{im}(\theta_{t^{-1}\mathbf{D}}(A)) = \text{Fil}_\mu^1 D_{\mathbb{C}_p} = D'_{\mathbb{C}_p} = \text{im} \begin{pmatrix} 0 \\ \text{Id}_{h_2} \end{pmatrix}$  in contradiction to  $\theta_{\mathbf{M}(1-h_1, h_1)}(A_1) \neq 0$ . This proves that  $\mu$  is weakly admissible.

5. Here the sub-isocrystals  $(0) \neq \underline{D}' \subsetneq \underline{D}$  are of two kinds. For the first kind  $t^{-1}\mathbf{D}'$  has rank 4 and is the kernel of a non-zero map  $E = (E_1|E_2|E_3) : t^{-1}\mathbf{D} \rightarrow \mathbf{M}(-1, 2)$  for  $E_i \in \text{End}_\varphi(\mathbf{M}(-1, 2))$ . Then  $t_N(\underline{D}') = 2$  and we claim that  $t_H(\underline{D}') \leq 2$ . Indeed

$$2 < t_H(\underline{D}') = \dim_{\mathbb{C}_p}(D'_{\mathbb{C}_p} \cap \text{Fil}_\mu^1 D_{\mathbb{C}_p}) \leq \dim_{\mathbb{C}_p} \text{Fil}_\mu^1 D_{\mathbb{C}_p} = 3$$

would imply that  $\text{Fil}_\mu^1 D_{\mathbb{C}_p} \subset D'_{\mathbb{C}_p}$  and  $\theta_{\mathbf{M}(-1, 2)}(EA) = 0$ . Hence  $EA = T \cdot B$  for some map  $B \in \text{Hom}_\varphi(\mathbf{M}(-1, 3), \mathbf{M}(1, 2)) = (0)$  and therefore  $E_1A_1 + E_2A_2 + E_3A_3 = 0$  in contradiction to the linear independence of  $A_1, A_2, A_3$  over  $\text{End}_\varphi(\mathbf{M}(-1, 2))$ .

For the second kind of sub-isocrystal  $t^{-1}\mathbf{D}' \cong \mathbf{M}(-1, 2)$  and  $t_N(\underline{D}') = 1$ . Again we claim that  $t_H(\underline{D}') \leq 1$ . Indeed

$$1 < t_H(\underline{D}') = \dim_{\mathbb{C}_p}(D'_{\mathbb{C}_p} \cap \text{Fil}_\mu^1 D_{\mathbb{C}_p}) \leq \dim_{\mathbb{C}_p} D'_{\mathbb{C}_p} = 2$$

implies  $D'_{\mathbb{C}_p} \subset \text{Fil}_\mu^1 D_{\mathbb{C}_p}$  and hence  $t^{-1}\mathbf{D}' \subset \mathbf{M}_\mu \subset t^{-1}\mathbf{D} = \mathbf{M}(-1, 2)^{\oplus 3}$ . Therefore  $\mathbf{M}_\mu \cong \mathbf{M}(-1, 2) \oplus \bigoplus_i \mathbf{M}(s_i, n_i)$  for  $-\frac{1}{2} \leq \frac{s_i}{n_i} \leq \frac{1}{2}$ . We had defined  $\mu$  by constructing a map  $\mathbf{M}(-1, 3) \rightarrow \mathbf{M}_\mu$ . If the induced map  $\mathbf{M}(-1, 3) \rightarrow \mathbf{M}(s_i, n_i)$  were non-zero for some  $i$ , then  $\frac{s_i}{n_i} \leq -\frac{1}{3}$  and  $s_i \leq -1, n_i \geq 2$ . The constraints on the remaining  $\frac{s_i}{n_i}$  imply that  $\deg \mathbf{M}_\mu \leq -1$ , a contradiction to Theorem 4.5 and Lemma 5.1. Hence  $A : \mathbf{M}(-1, 3) \rightarrow t^{-1}\mathbf{D}$  factors through  $t^{-1}\mathbf{D}'$ . But  $t^{-1}\mathbf{D}'$  lies in the kernel of an appropriate morphism  $0 \neq (E_1, E_2, E_3) : t^{-1}\mathbf{D} \rightarrow \mathbf{M}(-1, 2)$  for  $E_i \in \text{End}_\varphi(\mathbf{M}(-1, 2))$ . Again this means  $E_1A_1 + E_2A_2 + E_3A_3 = 0$  in contradiction to the linear independence of the  $A_i$ . This shows that  $t_H(\underline{D}') \leq t_N(\underline{D}')$  and so  $\mu$  is weakly admissible.

Therefore we have established in all cases that  $\mu \in \check{\mathcal{F}}_b^{wa} \setminus \check{\mathcal{F}}_b^0$  and the theorem is proved.  $\square$

## 9 Conjectures

As stated in Conjectures 5.4 and 5.5 we expect that the open subspace  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}_b^{wa}$  is arcwise connected and satisfies  $\check{\mathcal{F}}_b^0(L) = \check{\mathcal{F}}_b^{wa}(L)$  if the value group of  $L$  is finitely generated. This belief comes from the function field analog [Har10], [Har09, §§5.3 and 6.2] of the theory developed here. In that analog we were able to prove the assertions corresponding to these conjectures; see [Har10, Theorems 2.5.3 and 3.2.5].

Unfortunately we have little more to support Conjecture 5.4. However, the proof of Theorem 6.3 and the computations in Example 5.7 and Theorem 8.3 suggest that the fields  $L$  with  $\check{\mathcal{F}}_b^0(L) \subsetneq \check{\mathcal{F}}_b^{wa}(L)$  must contain elements of the form

$$\sum_{j=0}^{c-1} p^j \sum_{\nu \in \mathbb{Z}} p^{c\nu} u_j^{p^{-c\nu}}$$

for  $0 < c \leq e$  and  $u_j \in C$ ,  $|u_j| < 1$ . Maybe one could show that fields containing such elements cannot have finitely generated value group.

### Evidence for the connectedness of $\check{\mathcal{F}}_b^0$

We suggest a strategy to prove that  $\check{\mathcal{F}}_b^0$  is arcwise connected by noting that this follows from Conjecture 5.4.

**Proposition 9.1.** (a) *There exists an arcwise connected paracompact open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^{wa}$  of  $\check{\mathcal{F}}^{\text{an}}$  whose associated rigid analytic space is the period space  $(\check{\mathcal{F}}_b^{wa})^{\text{rig}}$ .*

(b) *If  $(G, b, \{\mu\})$  satisfies (5.1) and Conjecture 5.4 is true then  $\check{\mathcal{F}}_b^0$  is arcwise connected.*

(c) *If  $(G, b, \{\mu\})$  satisfies (5.1) and  $\check{\mathcal{F}}_b^0$  is connected then any open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^a$  with  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}_b^a \subset \check{\mathcal{F}}_b^{wa}$  is also arcwise connected.*

*Proof of Proposition 9.1.* (a) Since  $\check{\mathcal{F}}^{\text{an}}$  has a finite covering by affine spaces, all its open  $\check{E}$ -analytic subspaces are paracompact by Lemma A.3. From its construction in [RZ96, Proposition 1.34] the period space  $(\check{\mathcal{F}}_b^{wa})^{\text{rig}}$  possesses an admissible covering  $\{X_i^{\text{rig}}\}_{i \in \mathbb{N}}$  by admissible subsets  $X_i^{\text{rig}} \subset \check{\mathcal{F}}^{\text{rig}}$  such that  $\check{\mathcal{F}}^{\text{rig}} \setminus X_i^{\text{rig}}$  is a finite union  $\bigcup_j \text{Sp } B_{i,j}$  of affinoid subdomains and  $X_i^{\text{rig}} \subset X_{i+1}^{\text{rig}}$ . The  $X_i^{\text{rig}}$  correspond to open  $\check{E}$ -analytic subspaces  $X_i := \check{\mathcal{F}}^{\text{an}} \setminus \bigcup_j \mathcal{M}(B_{i,j})$  of  $\check{\mathcal{F}}^{\text{an}}$ . Moreover, it follows from [RZ96, Proposition 1.34] that the closure  $\bar{X}_i$  of  $X_i$  in  $\check{\mathcal{F}}^{\text{an}}$  is a finite union of  $\check{E}$ -affinoid subdomains and  $\bar{X}_i \subset X_{i+1}$ . Thus the union  $\check{\mathcal{F}}_b^{wa} := \bigcup_{i \in \mathbb{N}} X_i$  is an open  $\check{E}$ -analytic subspace of  $\check{\mathcal{F}}^{\text{an}}$  whose associated rigid analytic space equals  $(\check{\mathcal{F}}_b^{wa})^{\text{rig}}$ . (Use [Beh93, Lemma 1.6.2] to see that the Grothendieck topology induced from  $\check{\mathcal{F}}_b^{wa}$  on  $\{\mu \in \check{\mathcal{F}}_b^{wa} : \mathcal{H}(\mu)/\check{E} \text{ is finite}\}$  coincides with the Grothendieck topology of  $(\check{\mathcal{F}}_b^{wa})^{\text{rig}}$ .)

To prove the connectedness of  $\check{\mathcal{F}}_b^{wa}$  note that every  $\check{E}$ -analytic space is locally arcwise connected by [Beh90, Theorem 3.2.1]. The flag variety  $\check{\mathcal{F}}^{\text{an}}$  has a finite covering by affine spaces, hence a countable affinoid covering by polydiscs  $U \cong \mathcal{M}(\check{E}\langle \frac{x_1}{\zeta}, \dots, \frac{x_n}{\zeta} \rangle)$  for varying  $\zeta \in \check{E}$ . Since the points  $x \in \check{\mathcal{F}}_b^{wa}$  with  $\mathcal{H}(x)$  finite over  $\check{E}$  lie dense in  $\check{\mathcal{F}}_b^{wa}$  by [Beh90, Proposition 2.1.15] it suffices to exhibit for every such point  $x \in \check{\mathcal{F}}_b^{wa} \cap U$  a continuous map  $\alpha$  from the compact interval  $[0, |\zeta|]$  into  $\check{\mathcal{F}}_b^{wa} \cap U$  such that  $\alpha(0) = x$  and  $\alpha(|\zeta|)$  is the point corresponding to the Gauß norm on  $U$ .

So let  $x \in \check{\mathcal{F}}_b^{wa} \cap U$  with  $\mathcal{H}(x)$  finite over  $\check{E}$  be the point with coordinates  $x_i = c_i \in \mathcal{H}(x)$ ,  $|c_i| \leq |\zeta|$ . To define  $\alpha : [0, |\zeta|] \rightarrow \check{\mathcal{F}}_b^{wa} \cap U$  we expand each  $f \in \check{E}\langle \frac{x_1}{\zeta}, \dots, \frac{x_n}{\zeta} \rangle$  around  $x$

$$f = \sum_{\underline{i} \in \mathbb{N}_0^n} a_{\underline{i}} (x_1 - c_1)^{i_1} \cdots (x_n - c_n)^{i_n} \quad \text{with } a_{\underline{i}} \in \check{E}(c_1, \dots, c_n).$$

For every element  $t \in [0, |\zeta|]$  we obtain an analytic point  $P = P(x, t)$  in  $U$  by setting

$$|f|_P := \sup\{ |a_{\underline{i}}| t^{i_1 + \dots + i_n} : \underline{i} \in \mathbb{N}_0^n \}.$$

If  $x$  is fixed it is easy to see that this defines a continuous and injective map  $\alpha : [0, |\zeta|] \rightarrow U$ ,  $t \mapsto P(x, t)$  with  $\alpha(0) = x$  and  $\alpha(|\zeta|)$  being the point corresponding to the Gauß norm on  $U$ . It remains to show that  $\alpha(t)$  lies in  $\check{\mathcal{F}}_b^{wa}$ . For  $t > 0$  the prime ideal  $\ker |\cdot|_{\alpha(t)} := \{ f \in \check{E}\langle \frac{x_1}{\zeta}, \dots, \frac{x_n}{\zeta} \rangle : |f|_{\alpha(t)} = 0 \}$  corresponding to  $\alpha(t)$  is the zero ideal. Since by construction in [RZ96, Proposition 1.36] the set  $U \setminus \check{\mathcal{F}}_b^{wa}$  is a union of Zariski closed subsets of  $U$  and since it does not contain the point  $x$ , we obtain  $\alpha(t) \in \check{\mathcal{F}}_b^{wa}$ . This implies that  $\check{\mathcal{F}}_b^{wa}$  is arcwise connected.

(b) To prove the connectedness of  $\check{\mathcal{F}}_b^0$  we show that even  $\alpha([0, |\zeta|]) \subset \check{\mathcal{F}}_b^0 \cap U$ . Namely, the value group of  $\mathcal{H}(\alpha(t))$  is generated by  $|\check{E}(c_1, \dots, c_n)^\times|$  and  $t$  and hence is finitely generated. So Conjecture 5.4 will imply  $\alpha(t) \in \check{\mathcal{F}}_b^0 \cap U$  and so  $\check{\mathcal{F}}_b^0$  is arcwise connected.

(c) To prove the connectedness of  $\check{\mathcal{F}}_b^a$  we note that  $\check{\mathcal{F}}_b^a(L) = \check{\mathcal{F}}_b^{wa}(L)$  for all finite extensions  $L/\check{E}$  because  $\check{\mathcal{F}}_b^0(L) = \check{\mathcal{F}}_b^{wa}(L)$ . Since the points  $x \in \check{\mathcal{F}}_b^a$  with  $\mathcal{H}(x)$  finite over  $\check{E}$  lie dense in  $\check{\mathcal{F}}_b^a$  by [Beh90, Proposition 2.1.15],  $\check{\mathcal{F}}_b^0$  is a dense connected subset of  $\check{\mathcal{F}}_b^a$ .  $\square$

### Evidence for the maximality of $\check{\mathcal{F}}_b^0$

We conjectured in Conjecture 5.10 that the open  $\check{E}$ -analytic subspace  $\check{\mathcal{F}}_b^0 \subset \check{\mathcal{F}}_b^{wa}$  and the local system  $\mathcal{V}$  from Theorems 5.8 are maximal. For this it remains to prove that the local system does not extend to a larger open subspace  $\check{\mathcal{F}}_b^0 \subsetneq Y \subset \check{\mathcal{F}}_b^{wa}$ . This is supported by the following results of Andreatta and Brinon [And06, AB09]. Assume there exists a local system  $\mathcal{V}$  of  $\mathbb{Q}_p$ -vector spaces on  $Y$  as in Theorem 5.8 and let  $\tilde{Y}$  be the space of  $\mathbb{Z}_p$ -lattices inside  $\mathcal{V}$ .

Consider a morphism  $f : U \rightarrow \tilde{Y}$  of  $\check{E}$ -analytic spaces such that  $U$  possesses an affine formal model  $\mathrm{Spf} R$  that is étale over  $\mathrm{Spf} W(\mathbb{F}_p^{\mathrm{alg}})\langle T_1, T_1^{-1}, \dots, T_d, T_d^{-1} \rangle$ . (This hypothesis on  $U$  can even be weakened slightly; see [And06, AB09].) The pullback of  $\mathcal{V}$  to  $U$  corresponds by Proposition 1.2 to a representation  $\rho_V : \pi_1^{\mathrm{ét}}(U, \bar{x}) \rightarrow \mathrm{GL}(V)(\mathbb{Q}_p)$  which stabilizes a  $\mathbb{Z}_p$ -lattice in  $V = \mathcal{V}_{\bar{x}}$ . Hence  $\rho_V$  factors through  $\pi_1^{\mathrm{ét}}(U, \bar{x}) \rightarrow \pi_1^{\mathrm{alg}}(U, \bar{x}) \rightarrow \mathrm{GL}(V)(\mathbb{Z}_p) \subset \mathrm{GL}(V)(\mathbb{Q}_p)$ , where  $\pi_1^{\mathrm{alg}}(U, \bar{x})$  is the algebraic fundamental group classifying *finite* étale coverings. By [And06, Theorem 7.11]  $\rho_V$  corresponds to an étale  $(\varphi, \Gamma)$ -module  $\mathbf{M}$  over  $\mathbf{A}_R$  which is overconvergent by [AB09]. If  $\mu \in U$  is an analytic point and  $C$  is the completion of an algebraic closure of  $\mathcal{H}(\mu)$  the fiber  $\mathbf{M}_\mu \otimes \tilde{\mathbf{B}}(C)$  at  $\mu$  is therefore of the form  $\mathbf{M}_\mu^\dagger \otimes_{\tilde{\mathbf{B}}^\dagger(C)} \tilde{\mathbf{B}}(C)$  for a  $\varphi$ -module  $\mathbf{M}_\mu^\dagger$  over  $\tilde{\mathbf{B}}^\dagger(C)$ . By [Ked05, Proposition 5.5.1] the  $\varphi$ -module  $\mathbf{M}_\mu^\dagger \otimes_{\tilde{\mathbf{B}}^\dagger(C)} \tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C)$  over  $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C)$  is isomorphic to  $\mathbf{M}(0, 1)^{\oplus \dim V}$ . Due to (1.8) we have  $\mathbf{M}_\mu^\dagger \otimes_{\tilde{\mathbf{B}}^\dagger(C)} \tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger(C) \cong \mathbf{M}((V_{K_0}, b \cdot \varphi, \mathrm{Fil}_{f(\mu)}^\bullet V_{\mathcal{H}(\mu)}))$ . Thus the image of  $U$  in  $Y$  must be contained in  $\check{\mathcal{F}}_b^0$ .

However, at present this approach does not prove Conjecture 5.10 because unfortunately  $Y$  cannot be covered by  $\check{E}$ -analytic spaces  $U$  allowed by [And06, AB09]. For instance

consider on an annulus  $Y_1 = \mathcal{M}(\check{E}\langle T, \frac{p}{T} \rangle)$  the point  $y_1$  with  $|\sum_i a_i T^i|_y = \sup\{|a_i| |p|^{r_i} : i \in \mathbb{Z}\}$  for  $r \in [0, 1] \setminus \mathbb{Q}$ . Furthermore, consider a point  $y_2$  with  $\mathcal{H}(y_2) = \mathbb{C}_p$  in a small polydisc  $Y_2 = \mathcal{M}(\check{E}\langle \frac{T_1}{\zeta}, \dots, \frac{T_n}{\zeta} \rangle)$  for  $\zeta \in \check{E}$ ,  $0 < |\zeta| \ll 1$  (such points exist; see [Har10, Example A.2.2 (d)]). If  $Y$  contains  $Y_1 \times Y_2$  then the point  $y = (y_1, y_2) \in Y$  does not lie in the image of any  $U$  as above. So this method does not yield  $y \in \check{\mathcal{F}}_b^0$ . Also Conjecture 5.4 does not give  $y \in \check{\mathcal{F}}_b^0$  because the value group  $|\mathcal{H}(y)^\times| \supset |\mathcal{H}(y_2)^\times| \cong \mathbb{Q}$  is not finitely generated.

## A Berkovich's Analytic Spaces

Let  $\mathcal{O}_F$  be a (not necessarily discrete) complete valuation ring of rank one and let  $F$  be its fraction field. We briefly recall Berkovich's [Beh90, Beh93] theory of  $F$ -analytic spaces. Let  $B$  be an affinoid  $F$ -algebra in the sense of [BGR84, Chapter 6] with  $F$ -Banach norm  $|\cdot|$ . Berkovich calls these algebras *strictly  $F$ -affinoid*.

**Definition A.1.** An *analytic point*  $x$  of  $B$  is a semi-norm  $|\cdot|_x : B \rightarrow \mathbb{R}_{\geq 0}$  which satisfies:

- (a)  $|f + g|_x \leq \max\{|f|_x, |g|_x\}$  for all  $f, g \in B$ ,
- (b)  $|fg|_x = |f|_x |g|_x$  for all  $f, g \in B$ ,
- (c)  $|\lambda|_x = |\lambda|$  for all  $\lambda \in F$ ,
- (d)  $|\cdot|_x : B \rightarrow \mathbb{R}_{\geq 0}$  is continuous with respect to the norm  $|\cdot|$  on  $B$ .

The set of all analytic points of  $B$  is denoted  $\mathcal{M}(B)$ . On  $\mathcal{M}(B)$  one considers the coarsest topology such that for every  $f \in B$  the map  $\mathcal{M}(B) \rightarrow \mathbb{R}_{\geq 0}$  given by  $x \mapsto |f|_x$  is continuous. Equipped with this topology,  $\mathcal{M}(B)$  is a compact Hausdorff space; see [Beh90, Theorem 1.2.1]. Such a space is called a *strictly  $F$ -affinoid space*.

Every morphism  $\alpha : B \rightarrow B'$  of affinoid  $F$ -algebras is automatically continuous and hence induces a continuous morphism  $\mathcal{M}(\alpha) : \mathcal{M}(B') \rightarrow \mathcal{M}(B)$  by mapping the semi-norm  $B' \rightarrow \mathbb{R}_{\geq 0}$  to the composition  $B \rightarrow B' \rightarrow \mathbb{R}_{\geq 0}$ . By definition the  $\mathcal{M}(\alpha)$  are the *morphisms* in the category of strictly  $F$ -affinoid spaces. In particular, for an affinoid subdomain  $\mathrm{Sp} B' \subset \mathrm{Sp} B$  this morphism identifies  $\mathcal{M}(B')$  with a closed subset of  $\mathcal{M}(B)$ .

For every analytic point  $x \in \mathcal{M}(B)$  we let  $\ker |\cdot|_x := \{b \in B : |b|_x = 0\}$ . It is a prime ideal in  $B$ . We define the (*complete*) *residue field of  $x$*  as the completion with respect to  $|\cdot|_x$  of the fraction field of  $B/\ker |\cdot|_x$ . It will be denoted  $\mathcal{H}(x)$ . There is a natural continuous homomorphism  $B \rightarrow \mathcal{H}(x)$  of  $F$ -algebras. Conversely let  $L$  be a *complete extension* of  $F$ , by which we mean a field extension of  $F$  equipped with an absolute value  $|\cdot| : L \rightarrow \mathbb{R}_{\geq 0}$  which restricts on  $F$  to the norm of  $F$  such that  $L$  is complete with respect to  $|\cdot|$ . Any continuous  $F$ -algebra homomorphism  $B \rightarrow L$  defines on  $B$  a semi-norm which is an analytic point.

In [Beh93, §1.2] Berkovich defines the category of *strictly  $F$ -analytic spaces*. These spaces are topological spaces which admit an atlas with strictly  $F$ -affinoid charts.

The spaces  $\mathcal{M}(B)$  for affinoid  $F$ -algebras  $B$  are examples for strictly  $F$ -analytic spaces. Other examples arise from schemes  $Y$  which are locally of finite type over  $F$ , see [Beh90, §3.4]. Namely if  $Y = \mathbb{A}_F^n$  is affine  $n$ -space over  $F$  the associated strictly  $F$ -analytic space  $Y^{\mathrm{an}} = (\mathbb{A}_F^n)^{\mathrm{an}}$  consists of all semi-norms on the polynomial ring  $F[y_1, \dots, y_n]$  as in Definition A.1 (a)–(c). The topology on  $(\mathbb{A}_F^n)^{\mathrm{an}}$  is the coarsest topology such that for all  $f \in F[y_1, \dots, y_n]$  the map  $(\mathbb{A}_F^n)^{\mathrm{an}} \rightarrow \mathbb{R}_{\geq 0}, x \mapsto |f|_x$  is continuous. The space  $(\mathbb{A}_F^n)^{\mathrm{an}}$

is the union of the increasing sequence of compact polydiscs  $\mathcal{M}(F\langle \frac{y_1}{\zeta^m}, \dots, \frac{y_n}{\zeta^m} \rangle)$  of radii  $(|\zeta^m|, \dots, |\zeta^m|)$  for  $m \in \mathbb{N}_0$  where  $\zeta \in F$  has  $|\zeta| > 1$ . If  $Y \subset \mathbb{A}_F^n$  is a closed subscheme of affine  $n$ -space with coherent ideal sheaf  $\mathcal{I}$ , the ideal sheaf  $\mathcal{I}\mathcal{O}_{(\mathbb{A}_F^n)^{\text{an}}}$  defines a closed strictly  $F$ -analytic subspace  $Y^{\text{an}}$  of  $(\mathbb{A}_F^n)^{\text{an}}$ . Finally, if  $Y$  is arbitrary and  $\{Y_i\}_i$  is a covering of  $Y$  by affine open subschemes then one can glue the associated strictly  $F$ -affinoid spaces  $Y_i^{\text{an}}$  to the  $F$ -analytic space  $Y^{\text{an}}$ . Moreover  $Y^{\text{an}}$  is Hausdorff if and only if the scheme  $Y$  is separated, see [Beh90, Theorems 3.4.1 and 3.4.8].

The relation between strictly  $F$ -analytic spaces, rigid analytic spaces, and formal schemes is as follows. To every strictly  $F$ -analytic space  $X$  which is Hausdorff one can associate a quasi-separated rigid analytic space

$$X^{\text{rig}} := \{x \in X : \mathcal{H}(x) \text{ is a finite extension of } F\},$$

see [Beh93, §1.6]. Recall that a rigid analytic space is called *quasi-separated* if the intersection of any two affinoid subdomains is a finite union of affinoid subdomains. To describe the subcategories on which the functor  $X \mapsto X^{\text{rig}}$  is an equivalence we need the following terminology. A topological Hausdorff space is called *paracompact* if every open covering  $\{U_i\}_i$  has a locally finite refinement  $\{V_j\}_j$ , where *locally finite* means that every point has a neighborhood which meets only finitely many of the  $V_j$ . On the other hand an admissible covering of a rigid analytic space is said to be *of finite type* if every member of the covering meets only finitely many of the other members. A rigid analytic space over  $F$  is called *quasi-paracompact* if it possesses an admissible affinoid covering of finite type. Similarly we define the notions of *finite type* and *quasi-paracompact* also for (an open covering of) an admissible formal  $\mathcal{O}_F$ -scheme in the sense of Raynaud [Ray74]; see also [BL93].

**Theorem A.2.** *The following three categories are equivalent:*

- (a) *the category of paracompact strictly  $F$ -analytic spaces,*
- (b) *the category of quasi-separated quasi-paracompact rigid analytic spaces over  $F$ , and*
- (c) *the category of quasi-paracompact admissible formal  $\mathcal{O}_F$ -schemes, localized by admissible formal blowing-ups.*

*Proof.* It is shown in [Beh93, Theorem 1.6.1] that  $X \mapsto X^{\text{rig}}$  is an equivalence between (a) and (b). The equivalence of (c) with (b) is due to Raynaud. See [Bos05, Theorem 2.8/3] for a proof.  $\square$

Regarding paracompactness the following result was proved in [Har10, Lemma A.2.6].

**Lemma A.3.** *Let  $F$  be discretely valued with countable residue field and let  $X$  be a strictly  $F$ -analytic space. Assume that  $X$  is Hausdorff and admits a countable covering by strictly  $F$ -analytic spaces. Then every open subset of  $X$  is a paracompact strictly  $F$ -analytic space.*

This applies in particular if  $X = Y^{\text{an}}$  for a separated scheme  $Y$  of finite type over  $F$ .

We will also need the following well known fact which we were unable to find in the literature.

**Proposition A.4.** *Let  $f : X \rightarrow Y$  be an étale morphism of  $F$ -analytic spaces ([Beh93, §3.3]) such that  $f$  is bijective on  $C$ -valued points for any algebraically closed complete extension  $C$  of  $F$ . Then  $f$  is an isomorphism.*

*Proof.* Let  $x \in X$  and  $y = f(x) \in Y$ . Since  $f$  is quasi-finite there are open neighborhoods  $V = V_x \subset X$  of  $x$  and  $U = U_y \subset Y$  of  $y$  with  $f|_V : V \rightarrow U$  finite étale. Since  $f$  is open by [Beh93, Proposition 3.2.7] we may replace  $U$  by  $f(V)$ . For any affinoid  $\mathcal{M}(A) \subset U$  the preimage  $(f|_V)^{-1}(\mathcal{M}(A)) = \mathcal{M}(B)$  is affinoid with  $B$  finite flat over  $A$  by [Beh93, Proposition 3.2.3]. Hence  $B$  is finite locally free since  $A$  is noetherian by [Beh90, Proposition 2.1.3]. For any  $C$ -valued point of  $\mathcal{M}(A)$  the fiber  $\mathcal{M}(B \otimes_A C) \rightarrow \mathcal{M}(C)$  is finite étale by [Beh93, Corollary 3.3.8] and bijective by our assumption. Since  $C$  is algebraically closed,  $B \otimes_A C = C$  and so  $\text{rk}_A B = 1$ . Therefore the inverse map  $(f|_V)^{-1} : U \rightarrow V$  exists by [Beh93, Proposition 1.2.15(i)]. Now our assumption on bijectivity on  $C$ -valued points implies that the local inverses  $(f|_V)^{-1} : U \xrightarrow{\simeq} V$  glue by [Beh93, Proposition 1.3.2] to the global inverse of  $f$ .  $\square$

## References

- [And06] F. Andreatta: Generalized ring of norms and generalized  $(\varphi, \Gamma)$ -modules, *Ann. Sci. École Norm. Sup. (4)* **39** (2006), no. 4, 599–647.
- [AB09] F. Andreatta, O. Brinon: Surconvergence des représentations  $p$ -adiques: le cas relatif, in *Représentations  $p$ -adiques de groupes  $p$ -adiques I. représentations galoisiennes et  $(\varphi, \Gamma)$ -modules*, Astérisque **319** (2009), 39–116.
- [AB10] F. Andreatta, O. Brinon:  $\mathbf{B}_{\text{dR}}$ -représentations dans le cas relatif, *Ann. Sci. Éc. Norm. Supér. (4)* **43** (2010), no. 2, 279–339.
- [Ber02] L. Berger: Représentations  $p$ -adiques et équations différentielles, *Invent. Math.* **148** (2002), 219–284.
- [Ber04] L. Berger: Limites de représentations cristallines, *Compos. Math.* **140** (2004), no. 6, 1473–1498.
- [Ber09] L. Berger: Équations différentielles  $p$ -adiques et  $(\varphi, N)$ -modules filtrés, in *Représentations  $p$ -adiques de groupes  $p$ -adiques I : représentations galoisiennes et  $(\varphi, \Gamma)$ -modules*, Astérisque **319** (2009), 13–38; see also [arXiv:math.NT/0406601](https://arxiv.org/abs/math.NT/0406601).
- [Beh90] V.G. Berkovich: *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs **33**, Amer. Math. Soc., Providence 1990.
- [Beh93] V.G. Berkovich: Étale cohomology for non-Archimedean analytic spaces, *Publ. Math. I.H.E.S.* **78** (1993), 5–161.
- [Bos05] S. Bosch: *Lectures on Formal and Rigid Geometry*, Preprint **378**, University of Münster, SFB 478–Preprint Series, Münster 2005, available at <http://www.math.uni-muenster.de/sfb/about/publ/heft378.pdf>.
- [BGR84] S. Bosch, U. Güntzer, R. Remmert: *Non-archimedean analysis*, Grundlehren **261**, Springer-Verlag, Berlin-Heidelberg 1984.
- [BL93] S. Bosch, W. Lütkebohmert: Formal and Rigid Geometry  
I. Rigid Spaces, *Math. Ann.* **295** (1993), 291–317.  
II. Flattening Techniques, *Math. Ann.* **296** (1993), 403–429.
- [Bre00] C. Breuil: Groupes  $p$ -divisibles, groupes finis et modules filtrés, *Ann. of Math.* **152** (2000), 489–549.
- [Bri04] O. Brinon: *Représentations Galoisiennees  $p$ -adiques dans le cas relative*, Thèse de l’Université de Paris 11 (Orsay), 2004.
- [Col02] P. Colmez: Espaces de Banach de dimension finie, *J. Inst. Math. Jussieu* **1** (2002), 331–439.

- [Col09] P. Colmez: Espaces Vectoriels de dimension finie et représentations de de Rham, in *Représentations  $p$ -adiques de groupes  $p$ -adiques I: représentations galoisiennes et  $(\varphi, \Gamma)$ -modules*, Astérisque **319** (2009), 117–186.
- [CF00] P. Colmez, J.-M. Fontaine: Construction des représentations  $p$ -adiques semi-stables, *Invent. Math.* **140** (2000), no. 1, 1–43.
- [DOR] J.-F. Dat, S. Orlik, M. Rapoport: *Period domains over finite and  $p$ -adic fields*, forthcoming book.
- [DM82] P. Deligne, J. Milne: Tannakian Categories, in *Hodge Cycles, Motives, and Shimura Varieties*, pp. 101–228, LNM **900**, Springer-Verlag, New York 1982.
- [Fal99] G. Faltings: Integral crystalline cohomology over very ramified valuation rings, *J. Amer. Math. Soc.* **12** (1999), no. 1, 117–144.
- [Fal10] G. Faltings: Coverings of  $p$ -adic period domains, *J. Reine Angew. Math.* **643** (2010), 111–139.
- [Far04] L. Fargues: Cohomologie des espaces de modules de groupes  $p$ -divisibles et correspondances de Langlands locales, in *Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales*, Astérisque **291** (2004), 1–199.
- [Fon79] J.-M. Fontaine: Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, in *Journées de géométrie algébrique de Rennes (III)*, Astérisque **65** (1979), 3–80.
- [Fon82] J.-M. Fontaine: Sur certains types de représentations  $p$ -adiques du groupe de Galois d’un corps local; construction d’un anneau de Barsotti-Tate, *Ann. of Math. (2)* **115** (1982), no. 3, 529–577.
- [Fon94] J.-M. Fontaine: Le corps des périodes  $p$ -adiques, *Périodes  $p$ -adiques (Bures-sur-Yvette, 1988)*, Astérisque **223** (1994), 59–111.
- [FW79a] J.-M. Fontaine, J.-P. Wintenberger: Le “corps des normes” de certaines extensions algébriques de corps locaux, *C. R. Acad. Sci. Paris Sér. A-B* **288** (1979), no. 6, 367–370.
- [FW79b] J.-M. Fontaine, J.-P. Wintenberger: Extensions algébrique et corps des normes des extensions APF des corps locaux, *C. R. Acad. Sci. Paris Sér. A-B* **288** (1979), no. 8, 441–444.
- [Fou05] L. Fourquaux: *Logarithme de Perrin-Riou pour des extensions associées à un groupe de Lubin-Tate*, Thesis, Paris 6, 2005, available at <http://www.umpa.ens-lyon.fr/~fourquau/files/these.pdf>.
- [Gro71] A. Grothendieck: Groupes de Barsotti-Tate et cristaux, *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pp. 431–436, Gauthier-Villars, Paris, 1971.
- [Gro74] A. Grothendieck: *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Séminaire de Mathématiques Supérieures 45, Presses de l’Université de Montréal, Montréal 1974.
- [Har08] U. Hartl: On period spaces for  $p$ -divisible groups, *C. R. Math. Acad. Sci. Paris Ser. I* **346** (2008), 1123–1128; also available on [arXiv:0709.3444](https://arxiv.org/abs/0709.3444).
- [Har09] U. Hartl: A Dictionary between Fontaine-Theory and its Analogue in Equal Characteristic, *J. Number Th.* **129** (2009), 1734–1757; see also [arXiv:math.NT/0607182](https://arxiv.org/abs/math.NT/0607182).
- [Har10] U. Hartl: Period Spaces for Hodge Structures in Equal Characteristic, to appear in *Ann. of Math.* (2010); see also [arXiv:math.NT/0511686](https://arxiv.org/abs/math.NT/0511686).
- [HP04] U. Hartl, R. Pink: Vector bundles with a Frobenius structure on the punctured unit disc, *Comp. Math.* **140** n.3 (2004), 689–716.
- [dJ95] J. de Jong: Étale Fundamental groups of non-Archimedean analytic spaces, *Comp. Math.* **97** (1995), 89–118.

- [Ked05] K. Kedlaya: Slope filtrations revisited, *Doc. Math.* **10** (2005), 447–525.
- [Kis06] M. Kisin: Crystalline Representations and  $F$ -Crystals, in *Algebraic geometry and number theory*, pp. 459–496, Progress in Math. 253, Birkhäuser Boston, Boston, MA, 2006.
- [Kot85] R. Kottwitz: Isocrystals with additional structure, *Compositio Math.* **56** (1985), no. 2, 201–220.
- [Kot97] R. E. Kottwitz: Isocrystals with additional structure II, *Compositio Math.* **109** (1997), no. 3, 255–339.
- [Man63] Y.I. Manin: The theory of commutative formal groups over fields of finite characteristic, *Russian Math. Surveys* **18** No. 6 (1963), 1–83.
- [Mes72] W. Messing: *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, LNM **264**, Springer-Verlag, Berlin-New York, 1972.
- [Mil80] J. Milne: *Étale cohomology*, Princeton Mathematical Series 33, Princeton University Press, Princeton, N.J., 1980.
- [RZ96] M. Rapoport, T. Zink: *Period Spaces for  $p$ -divisible Groups*, Ann. Math. Stud. **141**, Princeton University Press, Princeton 1996.
- [Ray74] M. Raynaud: Géométrie analytique rigide d’après Tate, Kiehl, . . . , *Bul. Soc. Math. France, Mémoires* **39/40** (1974), 319 - 327.
- [Ser79] [Ser79] J.-P. Serre: Groupes algébriques associés aux modules de Hodge-Tate, *Journées de Géométrie Algébrique de Rennes (Rennes, 1978)*, Vol. III, pp. 155–188, Astérisque **65**, Soc. Math. France, Paris 1979.
- [Win83] J.-P. Wintenberger: Le corps des normes de certaines extensions infinies de corps locaux; applications, *Ann. Sci. École Norm. Sup. (4)* **16** (1983), no. 1, 59–89.
- [Win97] J.-P. Wintenberger: Propriétés du groupe tannakien des structures de Hodge  $p$ -adiques et torseur entre cohomologies cristalline et étale, *Ann. Inst. Fourier* **47** (1997), 1289–1334.

Urs Hartl  
University of Muenster  
Institute of Mathematics  
Einsteinstr. 62  
D – 48149 Muenster  
Germany  
<http://www.math.uni-muenster.de/u/urs.hartl/>