A Short Review on Local Shtukas and Divisible Local Anderson Modules

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Abstract

We review the analog of crystalline Dieudonné theory for p-divisible groups in the arithmetic of function fields from [HS19]. In our theory p-divisible groups are replaced by divisible local Anderson modules, and Dieudonné modules are replaced by local shtukas. We also explain their relation to global objects like Drinfeld modules and A-motives. We review the cohomology realizations of local shtukas and their comparison isomorphisms, and in the last section we explain how this yields the function field analog of Fontaine's theory of p-adic Galois representations.
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Contents

1 Introduction

The theory of p-adic Galois representations is concerned with the continuous represenations

$$\rho\colon\ \mathrm{Gal}(L^{\mathrm{alg}}/L)\ \longrightarrow\ \mathrm{GL}_r(\mathbb{Q}_p) \tag{1.1}$$

of the absolute Galois group $\mathrm{Gal}(L^{\mathrm{alg}}/L)$ of a finite field extension L of $\mathbb{Q}_p$. It started with Tate's introduction of *p-divisible groups* in [Tat66]. These are also called *Barsotti-Tate groups.* The Tate module T_pX of a p-divisible group X of height r over L induces Galois representations $V_pX := T_pX \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathrm{H}^1_{\text{ét}}(X,\mathbb{Q}_p) := \mathrm{Hom}_{\mathbb{Z}_p}(T_pX,\mathbb{Q}_p)$ as in (1.1). If X extends to a p-divisible group over $\mathcal{O}_L$, one says that X has *good reduction.* In this case the special fiber $X_0 := X \otimes_{\mathcal{O}_L} \kappa$ of X over the residue field κ of $\mathcal{O}_L$ can be described by its *crystalline cohomology* $H^1_{\mathrm{cris}}\big(X_0/W(\kappa)\big)$, where $W(\kappa)$ is the ring of p-typical Witt vectors with coefficients in κ. The p-divisible group X, which can be viewed as a lift of X_0 to $\mathcal{O}_L$, is described by the F-crystal $H^1_{\mathrm{cris}}\big(X_0/W(\kappa)\big)$ together with its Hodge-filtration. All this was proved by Messing [Mes72]. Grothendieck [Gro74] reformulated this as a functor relating the p-adic étale cohomology $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p)$ to the crystalline cohomology $\mathrm{H}^1_{\text{cris}}(X_0/L_0)$ with its Hodge filtration, where $L_0 := W(\kappa)[\frac{1}{p}]$ and $\mathrm{H}^1_{\text{cris}}(X_0/L_0)$ is a *filtered isocrystal*; see Remark 6.6 below. Grothendieck then posed the problem to extend this functor, which he called the *mysterious functor*, to general proper smooth schemes X over L with good reduction. For those X the problem was solved by Fontaine [Fon79, Fon82, Fon90, Fon94], who defined the notion of *crystalline p-adic Galois representations* and constructed a functor from crystalline p-adic Galois representations to filtered isocrystals. Fontaine conjectured that $\mathrm{H}^i_{\text{ét}}(X \times_L L^{\text{alg}}, \mathbb{Q}_p)$ is crystalline when X is a proper smooth scheme over $\mathcal{O}_L$. After contributions by Grothendieck, Tate, Fontaine, Lafaille, Messing, Hyodo, Kato and many others, Fontaine's conjecture was proved independently by Faltings [Fal89], Niziol [Niz98] and Tsuji [Tsu99].

Our goal in this survey is to describe the function field analog of the above. In this analog, p-divisible groups are replaced by *divisible local Anderson modules* which we discuss in Section 4. The analog of Messing's [Mes72] theory of crystalline Dieudonné-modules for p-divisible groups is Theorem 4.2. In it Messings F-crystals are replaced by *local shtukas*, which we treat first in Section 2. The anti-equivalence between divisible local Anderson modules and local shtukas passes through finite flat group schemes and finite shtukas. We review it in Section 3. Analogous to the étale and crystalline cohomology we mentioned for p-divisible groups in the previous paragraph, local shtukas possess cohomology realizations as described in Section 5. In the final Section 6 we explain how the theory of local shtukas provides the function field analog of Fontaine's theory of p-adic Galois representations (1.1).

2 Local Shtukas

The theory of local shtukas is the function field analog of Fontaine's theory of p-adic Galois representations. Let A_ε be a complete discrete valuation ring with finite residue field $\mathbb{F}_\varepsilon$ of characteristic p such that the fraction field Q_ε of A_ε also has characteristic p. The rings A_ε and Q_ε are the function field analogs of $\mathbb{Z}_p$ and $\mathbb{Q}_p$. We choose a uniformizing parameter $z \in A_\varepsilon$. Then A_ε is canonically isomorphic to $\mathbb{F}_\varepsilon[\![z]\!]$. Let $\hat{q} = \#\mathbb{F}_\varepsilon$ be the cardinality of $\mathbb{F}_\varepsilon$. As base rings R over which our objects are defined we are interested in this article in two kinds of A_ε-algebras:

(a) The first kind are A_ε-algebras in which the image ζ of the uniformizer z of A_ε is nilpotent. We denote the category of these A_ε-algebras by $\mathcal{N}ilp_{A_\varepsilon}$.

(b) Let K be a field which is complete with respect to a non-trivial, non-archimedean absolute value $|\,.\,|\colon K \to \mathbb{R}_{\geq 0}$ and let $\mathcal{O}_K = \{x \in K \colon |x| \leq 1\}$ be the valuation ring of K. We make $\mathcal{O}_K$ into an A_ε-algebra via an *injective* ring homomorphism $\gamma\colon A_\varepsilon \hookrightarrow \mathcal{O}_K$ such that $\zeta := \gamma(z) \neq 0$ lies in the maximal ideal $\mathfrak{m}_K \subset \mathcal{O}_K$.

The relation between the two kinds of base rings is that $\mathcal{O}_K/(\zeta^n) \in \mathcal{N}ilp_{A_\varepsilon}$ for all positive integers n.

Let R be a base ring as in (a) or (b). To define local shtukas over R we consider modules $\hat{M}$ over the power series ring $R[\![z]\!]$, which Zariski locally on $\operatorname{Spec} R$ are free over $R[\![z]\!]$. We call such a module a *locally free $R[\![z]\!]$-module of rank r*. We set $\hat{M}[\frac{1}{z-\zeta}] := \hat{M} \otimes_{R[\![z]\!]} R[\![z]\!][\frac{1}{z-\zeta}]$, and $\hat{M}[\frac{1}{z}] := \hat{M} \otimes_{R[\![z]\!]} R(\!(z)\!)$ where $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$, and $\hat{\sigma}^*\hat{M} := \hat{M} \otimes_{R[\![z]\!],\hat{\sigma}} R[\![z]\!]$ where $\hat{\sigma}$ is the endomorphism of $R[\![z]\!]$ with $\hat{\sigma}(z) = z$ and $\hat{\sigma}(b) = b^{\hat{q}}$ for $b \in R$. Note that $R[\![z]\!][\frac{1}{z-\zeta}] = R(\!(z)\!)$ if $R \in \mathcal{N}ilp_{A_\varepsilon}$ as in (a), but $R[\![z]\!][\frac{1}{z-\zeta}] \neq R(\!(z)\!)$ if R is a valuation ring as in (b). There is a natural $\hat{\sigma}$-semilinear map $\hat{M} \to \hat{\sigma}^*\hat{M}$, $m \mapsto \hat{\sigma}^*_{\hat{M}} m := m \otimes 1$. For a morphism of $R[\![z]\!]$-modules $f\colon \hat{M} \to \hat{M}'$ we set $\hat{\sigma}^* f := f \otimes \mathrm{id}\colon \hat{\sigma}^*\hat{M} \to \hat{\sigma}^*\hat{M}'$.

Definition 2.1. A *local $\hat{\sigma}$-shtuka* (or *local shtuka*) *of rank r* over R is a pair $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ consisting of a locally free $R[\![z]\!]$-module $\hat{M}$ of rank r, and an isomorphism $\tau_{\hat{M}}\colon \hat{\sigma}^*\hat{M}[\frac{1}{z-\zeta}] \xrightarrow{\sim} \hat{M}[\frac{1}{z-\zeta}]$. If $\tau_{\hat{M}}(\hat{\sigma}^*\hat{M}) \subset \hat{M}$ then $\underline{\hat{M}}$ is called *effective*, and if $\tau_{\hat{M}}(\hat{\sigma}^*\hat{M}) = \hat{M}$ then $\underline{\hat{M}}$ is called *étale*. We say that $\tau_{\hat{M}}$ is *topologically nilpotent*, if $\underline{\hat{M}}$ is effective and there is an integer n such that $\operatorname{im}(\tau^n_{\hat{M}}) \subset z\hat{M}$, where $\tau^n_{\hat{M}} := \tau_{\hat{M}} \circ \hat{\sigma}^*\tau_{\hat{M}} \circ \ldots \circ \hat{\sigma}^{(n-1)*}\tau_{\hat{M}}\colon \hat{\sigma}^{n*}\hat{M} \to \hat{M}$.

A *morphism* of local shtukas $f\colon (\hat{M}, \tau_{\hat{M}}) \to (\hat{M}', \tau_{\hat{M}'})$ over R is a morphism of $R[\![z]\!]$-modules $f\colon \hat{M} \to \hat{M}'$ which satisfies $\tau_{\hat{M}'} \circ \hat{\sigma}^* f = f \circ \tau_{\hat{M}}$. We denote the set of morphisms from $\underline{\hat{M}}$ to $\underline{\hat{M}}'$ by $\mathrm{Hom}_R(\underline{\hat{M}}, \underline{\hat{M}}')$.

A *quasi-morphism* between local shtukas $f\colon (\hat{M}, \tau_{\hat{M}}) \to (\hat{M}', \tau_{\hat{M}'})$ over R is a morphism of $R(\!(z)\!)$-modules $f\colon \hat{M}[\frac{1}{z}] \to \hat{M}'[\frac{1}{z}]$ with $\tau_{\hat{M}'} \circ \hat{\sigma}^* f = f \circ \tau_{\hat{M}}$. It is called a *quasi-isogeny* if it is an isomorphism of $R(\!(z)\!)$-modules. We denote the set of quasi-morphisms from $\underline{\hat{M}}$ to $\underline{\hat{M}}'$ by $\mathrm{QHom}_R(\underline{\hat{M}}, \underline{\hat{M}}')$.

For any local shtuka $(\hat{M}, \tau_{\hat{M}})$ over $R \in \mathcal{N}ilp_{A_\varepsilon}$ the homomorphism $\hat{M} \to \hat{M}[\frac{1}{z-\zeta}]$ is injective by the flatness of $\hat{M}$ and the following

Lemma 2.2 ([HS19, Lemma 2.2]). *Let R be an A_ε-algebra as in (a) or (b). Then the sequence of $R[\![z]\!]$-modules*

$$0 \longrightarrow R[\![z]\!] \longrightarrow R[\![z]\!] \longrightarrow R \longrightarrow 0$$
$$1 \longmapsto z-\zeta \ , \quad z \longmapsto \zeta$$

is exact. In particular $R[\![z]\!] \subset R[\![z]\!][\frac{1}{z-\zeta}]$.

Of fundamental importance is the following

Example 2.3. Let $\mathbb{F}_q$ be a finite field with q elements, let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$, and let $Q := \mathbb{F}_q(C)$ be the function field of C. Fix a closed point ∞ of C, and let $A := \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$ be the ring of regular functions on C outside ∞. The rings A and Q are the function field analogs of $\mathbb{Z}$ and $\mathbb{Q}$.

Let $\varepsilon \subset A$ be a maximal ideal and let A_ε be the completion of A at ε. Then $\mathbb{F}_\varepsilon$ is a field extension of $\mathbb{F}_q$ with $\hat{q} := \#\mathbb{F}_\varepsilon = q^{[\mathbb{F}_\varepsilon:\mathbb{F}_q]}$. Let R be a base A_ε-algebra as in (a) or (b) and denote its structure morphism by $\gamma\colon A_\varepsilon \to R$. Set $A_R := A \otimes_{\mathbb{F}_q} R$ and let $\sigma := \mathrm{id}_A \otimes \mathrm{Frob}_{q,R}$ be the endomorphism of A_R with $\sigma(a \otimes b) = a \otimes b^q$ for $a \in A$ and $b \in R$. An *effective A-motive of rank r over R* is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_R-module M of rank r and an injective A_R-homomorphism $\tau_M\colon \sigma^* M \hookrightarrow M$ whose cokernel is a finite free R-module and annihilated by a power of the ideal $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a) \colon a \in A) = \ker(\gamma \otimes \mathrm{id}_R\colon A_R \twoheadrightarrow R) \subset A_R$.

More generally, an *A-motive of rank r over R* is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_R-module M of rank r and an isomorphism $\tau_M\colon \sigma^* M|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} \xrightarrow{\sim} M|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})}$ of the associated sheaves outside $\mathrm{V}(\mathcal{J}) \subset \mathrm{Spec}\, A_R$. Note that if $A = \mathbb{F}_q[t]$ then $\mathcal{J} = (t - \gamma(t))$ and $\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J}) = \mathrm{Spec}\, R[t][\frac{1}{t-\gamma(t)}]$.

Let $\underline{M}$ be an (effective) A-motive over R. We consider the ε-adic completions $A_{\varepsilon,R} = \varprojlim A_R/\varepsilon^n A_R$ of A_R and $\underline{M} \otimes_{A_R} A_{\varepsilon,R} := (M \otimes_{A_R} A_{\varepsilon,R}\,,\, \tau_M \otimes \mathrm{id})$ of $\underline{M}$. If $\mathbb{F}_\varepsilon = \mathbb{F}_q$, and hence $\hat{q} = q$ and $\hat{\sigma} = \sigma$, we have $A_{\varepsilon,R} = R[\![z]\!]$ and $\mathcal{J} \cdot A_{\varepsilon,R} = (z - \zeta)$ because $R \otimes_{A_R} A_{\varepsilon,R} = R$. So $\underline{M} \otimes_{A_R} A_{\varepsilon,R}$ is an (effective) local shtuka over R which we denote by $\underline{\hat{M}}_\varepsilon(\underline{M})$ and call the *local $\hat{\sigma}$-shtuka at ε associated with $\underline{M}$*. If $f := [\mathbb{F}_\varepsilon : \mathbb{F}_q] > 1$ the construction is slightly more complicated; compare the discussion in [BH11, after Proposition 8.4]. Namely, we consider the canonical isomorphism $\mathbb{F}_\varepsilon[\![z]\!] \xrightarrow{\sim} A_\varepsilon$ and the ideals $\mathfrak{a}_i = (a \otimes 1 - 1 \otimes \gamma(a)^{q^i} \colon a \in \mathbb{F}_\varepsilon) \subset A_{\varepsilon,R}$ for $i \in \mathbb{Z}/f\mathbb{Z}$, which satisfy $\prod_{i \in \mathbb{Z}/f\mathbb{Z}} \mathfrak{a}_i = (0)$, because $\prod_{i \in \mathbb{Z}/f\mathbb{Z}} (X - a^{q^i}) \in \mathbb{F}_q[X]$ is a multiple of the minimal polynomial of a over $\mathbb{F}_q$ and even equal to it when $\mathbb{F}_\varepsilon = \mathbb{F}_q(a)$. By the Chinese remainder theorem $A_{\varepsilon,R}$ decomposes

$$A_{\varepsilon,R} \;=\; \prod_{i \in \mathbb{Z}/f\mathbb{Z}} A_{\varepsilon,R}/\mathfrak{a}_i\,. \tag{2.1}$$

Each factor is canonically isomorphic to $R[\![z]\!]$. The factors are cyclically permuted by σ because $\sigma(\mathfrak{a}_i) = \mathfrak{a}_{i+1}$. In particular σ^f stabilizes each factor. The ideal $\mathcal{J}$ decomposes as follows $\mathcal{J} \cdot A_{\varepsilon,R}/\mathfrak{a}_0 = (z - \zeta)$ and $\mathcal{J} \cdot A_{\varepsilon,R}/\mathfrak{a}_i = (1)$ for $i \neq 0$. We define the *local $\hat{\sigma}$-shtuka at ε associated with $\underline{M}$* as $\underline{\hat{M}}_\varepsilon(\underline{M}) := (\hat{M}, \tau_{\hat{M}}) := \big(M \otimes_{A_R} A_{\varepsilon,R}/\mathfrak{a}_0\,,\, (\tau_M \otimes 1)^f\big)$, where $\tau_M^f := \tau_M \circ \sigma^* \tau_M \circ \ldots \circ \sigma^{(f-1)*} \tau_M$. Of course if $f = 1$ we get back the definition of $\underline{\hat{M}}_\varepsilon(\underline{M})$ given above. Also note if $\underline{M}$ is effective, then $M/\tau_M(\sigma^* M) = \hat{M}/\tau_{\hat{M}}(\hat{\sigma}^* \hat{M})$.

The local shtuka $\underline{\hat{M}}_\varepsilon(\underline{M})$ allows to recover $\underline{M} \otimes_{A_R} A_{\varepsilon,R}$ via the isomorphism

$$\bigoplus_{i=0}^{f-1} (\tau_M \otimes 1)^i \bmod \mathfrak{a}_i \colon \Big(\bigoplus_{i=0}^{f-1} \sigma^{i*}(M \otimes_{A_R} A_{\varepsilon,R}/\mathfrak{a}_0),\ (\tau_M \otimes 1)^f \oplus \bigoplus_{i\neq 0} \mathrm{id} \Big) \xrightarrow{\ \sim\ } \underline{M} \otimes_{A_R} A_{\varepsilon,R}\,,$$

because for $i \neq 0$ the equality $\mathcal{J} \cdot A_{\varepsilon,R}/\mathfrak{a}_i = (1)$ implies that $\tau_M \otimes 1$ is an isomorphism modulo $\mathfrak{a}_i$; see [BH11, Propositions 8.8 and 8.5] for more details. Note that $\underline{M} \mapsto \underline{\hat{M}}_\varepsilon(\underline{M})$ is a functor.

We quote the next lemma from [HS19, Lemma 2.3].

Lemma 2.4. *Let $(\hat{M}, \tau_{\hat{M}})$ be a local shtuka over R. Then there are $e, e' \in \mathbb{Z}$ such that $(z-\zeta)^{e'}\hat{M} \subset \tau_{\hat{M}}(\hat{\sigma}^*\hat{M}) \subset (z-\zeta)^{-e}\hat{M}$. For any such e the map $\tau_{\hat{M}} \colon \hat{\sigma}^*\hat{M} \to (z-\zeta)^{-e}\hat{M}$ is injective, and the quotient $(z-\zeta)^{-e}\hat{M}/\tau_M(\hat{\sigma}^*\hat{M})$ is a locally free R-module of finite rank.*

Example 2.5. We discuss the case of the Carlitz module [Car35]. We keep the notation from Example 2.3 and set $A = \mathbb{F}_q[t]$. Let $\mathbb{F}_q(\theta)$ be the rational function field in the variable θ and let $\gamma \colon A \to \mathbb{F}_q(\theta)$ be given by $\gamma(t) = \theta$. The *Carlitz motive over* $\mathbb{F}_q(\theta)$ is the A-motive $\underline{M} = \big(\mathbb{F}_q(\theta)[t], t-\theta\big)$.

Now let $\varepsilon = (z) \subset A$ be a maximal ideal generated by a monic prime element $z = z(t) \in \mathbb{F}_q[t]$. Then $\mathbb{F}_\varepsilon = A/(z)$ and A_ε is canonically isomorphic to $\mathbb{F}_\varepsilon[\![z]\!]$. Let $\mathcal{O}_K \supset \mathbb{F}_\varepsilon[\![\zeta]\!]$ be a valuation ring as in (b) and let $\theta = \gamma(t) \in \mathcal{O}_K$. The Carlitz motive has *good reduction* in the sense that it has a model over $\mathcal{O}_K$ given by the A-motive $\underline{M} = (\mathcal{O}_K[t], t-\theta)$ over $\mathcal{O}_K$.

If $\deg_t z(t) = 1$, that is $z(t) = t-a$ for $a \in \mathbb{F}_q$, then $\mathbb{F}_\varepsilon = \mathbb{F}_q$, $\zeta = \theta - a$, and $z - \zeta = t - \theta$. So $\underline{\hat{M}}_\varepsilon(\underline{M}) = (\mathcal{O}_K[\![z]\!], z-\zeta)$.

If $\deg_t z(t) = f > 1$, then $\underline{\hat{M}}_\varepsilon(\underline{M}) = \big(\mathcal{O}_K[\![z]\!], (t-\theta)(t-\theta^q)\cdots(t-\theta^{q^{f-1}})\big)$. Here the product $(t-\theta)(t-\theta^q)\cdots(t-\theta^{q^{f-1}}) = (z-\zeta)u$ for a unit $u \in \mathbb{F}_\varepsilon[\![\zeta]\!][\![z]\!]^\times$, because $\tau_M(\sigma^*M) = (t-\theta)M$ implies that $\underline{\hat{M}}_\varepsilon(\underline{M})$ is effective and $\hat{M}/\tau_{\hat{M}}(\hat{\sigma}^*\hat{M}) = M/\tau_M(\sigma^*M)$ is free over $\mathcal{O}_K$ of rank 1. In order to get rid of u we denote the image of t in $\mathbb{F}_\varepsilon$ by λ. Then $\mathbb{F}_\varepsilon = \mathbb{F}_q(\lambda)$ and $z(t)$ equals the minimal polynomial $(t-\lambda)\cdots(t-\lambda^{q^{f-1}})$ of λ over $\mathbb{F}_q$. Moreover, $t \equiv \lambda \bmod zA_\varepsilon$ and $\theta \equiv \lambda \bmod \zeta\mathbb{F}_\varepsilon[\![\zeta]\!]$. We compute in $\mathbb{F}_\varepsilon[\![\zeta]\!][\![z]\!]/(\zeta)$

$$z(t) \;=\; (t-\lambda)\cdots(t-\lambda^{q^{f-1}}) \;\equiv\; (t-\theta)\cdots(t-\theta^{q^{f-1}}) \;\equiv\; (z-\zeta)u \;\equiv\; zu \mod \zeta\,.$$

Since z is a non-zero-divisor in $\mathbb{F}_\varepsilon[\![\zeta]\!][\![z]\!]/(\zeta)$ it follows that $u \equiv 1 \bmod \zeta\,\mathbb{F}_\varepsilon[\![\zeta]\!][\![z]\!]$. We write $u = 1 + \zeta u'$ and observe that the product

$$w \;:=\; \prod_{n=0}^{\infty} \hat{\sigma}^n(u) \;=\; \prod_{n=0}^{\infty} \hat{\sigma}^n(1+\zeta u') \;=\; \prod_{n=0}^{\infty} \big(1 + \zeta^{\hat{q}^n}\hat{\sigma}^n(u')\big)$$

converges in $\mathbb{F}_\varepsilon[\![\zeta]\!][\![z]\!]^\times$ because $\mathbb{F}_\varepsilon[\![\zeta]\!][\![z]\!]$ is ζ-adically complete. It satisfies $w = u \cdot \hat{\sigma}(w)$ and so multiplication with w defines a canonical isomorphism $(\mathcal{O}_K[\![z]\!], z-\zeta) \xrightarrow{\ \sim\ } \underline{\hat{M}}_\varepsilon(\underline{M})$.

We conclude that $\underline{\hat{M}}_\varepsilon(\underline{M}) = (\mathcal{O}_K[\![z]\!], z-\zeta)$, regardless of $\deg_t z(t)$.

3 Finite Shtukas

In this section let R be an arbitrary $\mathbb{F}_\varepsilon$-algebra. For an R-module $\hat{M}$ we set $\hat{\sigma}^*\hat{M} := \hat{M} \otimes_{R,\,\mathrm{Frob}_{\hat{q}}} R$ where $\mathrm{Frob}_{\hat{q}}$ is the $\hat{q}$-Frobenius endomorphism of R with $\mathrm{Frob}_{\hat{q}}(b) = b^{\hat{q}}$ for $b \in R$.

Definition 3.1. A *finite $\mathbb{F}_\varepsilon$-shtuka* over R is a pair $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ consisting of a locally free R-module $\hat{M}$ of finite rank denoted $\mathrm{rk}\,\underline{\hat{M}}$, and an R-module homomorphism $\tau_{\hat{M}} \colon \hat{\sigma}^*\hat{M} \to \hat{M}$ satisfying

$f \circ \tau_{\hat{M}} = \tau_{\hat{M}'} \circ \hat{\sigma}^* f$. That is, the following diagram is commutative

$$\begin{array}{ccc} \hat{\sigma}^*\hat{M} & \xrightarrow{\hat{\sigma}^* f} & \hat{\sigma}^*\hat{M}' \\ \downarrow{\scriptstyle \tau_{\hat{M}}} & & \downarrow{\scriptstyle \tau_{\hat{M}'}} \\ \hat{M} & \xrightarrow{\quad f \quad} & \hat{M}' . \end{array}$$

A finite $\mathbb{F}_\varepsilon$-shtuka over R is called *étale* if $\tau_{\hat{M}}$ is an isomorphism. We say that $\tau_{\hat{M}}$ is *nilpotent* if there is an integer n such that $\tau^n_{\hat{M}} := \tau_{\hat{M}} \circ \hat{\sigma}^*\tau_{\hat{M}} \circ \ldots \circ \sigma^*_{q^{n-1}}\tau_{\hat{M}} = 0$.

Finite $\mathbb{F}_\varepsilon$-shtukas were studied at various places in the literature. They were called "(finite) φ-sheaves" by Drinfeld [Dri87, § 2], Taguchi and Wan [Tag95, TW96] and "Dieudonné $\mathbb{F}_q$-modules" by Laumon [Lau96]. Finite $\mathbb{F}_\varepsilon$-shtukas over a field admit a canonical decomposition.

Proposition 3.2. *([Lau96, Lemma B.3.10]) If $R = L$ is a field, every finite $\mathbb{F}_\varepsilon$-shtuka $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ is canonically an extension of finite $\mathbb{F}_\varepsilon$-shtukas*

$$0 \longrightarrow (\hat{M}_{\text{ét}}, \tau_{\text{ét}}) \longrightarrow (\hat{M}, \tau_{\hat{M}}) \longrightarrow (\hat{M}_{\text{nil}}, \tau_{\text{nil}}) \longrightarrow 0$$

where $\tau_{\text{ét}}$ is an isomorphism and τ_{nil} is nilpotent. $\underline{\hat{M}}_{\text{ét}} = (\hat{M}_{\text{ét}}, \tau_{\text{ét}})$ is the largest étale finite $\mathbb{F}_q$-sub-shtuka of $\underline{\hat{M}}$ and equals $\operatorname{im}(\tau^{\operatorname{rk}\hat{M}}_{\hat{M}})$. If L is perfect this extension splits canonically.

Example 3.3. Every effective local shtuka $(\hat{M}, \tau_{\hat{M}})$ of rank r over R yields for every $n \in \mathbb{N}$ a finite $\mathbb{F}_\varepsilon$-shtuka $\big(\hat{M}/z^n\hat{M}, \tau_{\hat{M}} \bmod z^n\big)$ of rank rn, and $(\hat{M}, \tau_{\hat{M}})$ equals the projective limit of these finite $\mathbb{F}_\varepsilon$-shtukas.

Thus from Proposition 3.2 we obtain

Proposition 3.4. *If $R = L$ is a field in $\mathcal{N}ilp_{A_\varepsilon}$, that is, $\zeta = 0$ in L, then every effective local shtuka $(\hat{M}, \tau_{\hat{M}})$ is canonically an extension of effective local shtukas*

$$0 \longrightarrow (\hat{M}_{\text{ét}}, \tau_{\text{ét}}) \longrightarrow (\hat{M}, \tau_{\hat{M}}) \longrightarrow (\hat{M}_{\text{nil}}, \tau_{\text{nil}}) \longrightarrow 0$$

where $\tau_{\text{ét}}$ is an isomorphism and τ_{nil} is topologically nilpotent. $(\hat{M}_{\text{ét}}, \tau_{\text{ét}})$ is the largest étale effective local sub-shtuka of $(\hat{M}, \tau_{\hat{M}})$. If L is perfect this extension splits canonically. □

Finite $\mathbb{F}_\varepsilon$-shtukas and local shtukas are related to group schemes in the following way. Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ be a finite $\mathbb{F}_\varepsilon$-shtuka over R. Let

$$E \;=\; \operatorname{Spec} \bigoplus_{n \geq 0} \operatorname{Sym}^n_R \hat{M}$$

be the geometric vector bundle corresponding to $\hat{M}$ over $\operatorname{Spec} R$, and let $F_{\hat{q},E} \colon E \to \hat{\sigma}^*E$ be its relative $\hat{q}$-Frobenius morphism over R. On the other hand, the map $\tau_{\hat{M}}$ induces another R-morphism $\operatorname{Spec}(\operatorname{Sym}^\bullet \tau_{\hat{M}}) \colon E \to \hat{\sigma}^*E$. Drinfeld defines

$$\operatorname{Dr}_{\hat{q}}(\underline{\hat{M}}) \;:=\; \ker\big(\operatorname{Spec}(\operatorname{Sym}^\bullet \tau_{\hat{M}}) - F_{\hat{q},E} \colon E \to \hat{\sigma}^*E\big) \;=\; \operatorname{Spec}\,\big(\bigoplus_{n \geq 0} \operatorname{Sym}^n_R \hat{M}\big)/I$$

where the ideal I is generated by the elements $m^{\otimes q} - \tau_{\hat{M}}(\hat{\sigma}^* m)$ for all elements m of $\hat{M}$. (Here $m^{\otimes q}$ lives in $\operatorname{Sym}^q_R \hat{M}$ and $\tau_{\hat{M}}(\hat{\sigma}^*m)$ in $\operatorname{Sym}^1_R \hat{M}$.) Note that locally on $\operatorname{Spec} R$ we have $\hat{M} = \bigoplus_{i=1}^d R \cdot m_i$ and $E \cong \operatorname{Spec} R[m_1, \ldots, m_d] = \mathbb{G}^d_{a,R}$. The subgroup scheme $\operatorname{Dr}_{\hat{q}}(\underline{\hat{M}})$ is finite locally free over R of order

$\hat{q}^{\operatorname{rk}\underline{\hat{M}}}$, that is, the R-algebra $\mathcal{O}_{\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})}$ is a finite locally free R-module of rank $\hat{q}^{\operatorname{rk}\underline{\hat{M}}}$. It is also an $\mathbb{F}_\varepsilon$-module scheme over R via the comultiplication $\Delta\colon m \mapsto m \otimes 1 + 1 \otimes m$ and the $\mathbb{F}_\varepsilon$-action $[a]\colon m \mapsto am$ which it inherits from E. It is even a *strict* $\mathbb{F}_\varepsilon$*-module scheme* in the sense of Faltings [Fal02] and Abrashkin [Abr06]. For a proof see [Abr06, Theorem 2] or [HS19, § 5]. This means that $\mathbb{F}_\varepsilon$ acts on the *co-Lie complex of* $\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$ *over* R, see Illusie [Ill72, § VII.3.1], via the scalar multiplication through $\mathbb{F}_\varepsilon \subset R$. A detailed explanation of strict $\mathbb{F}_\varepsilon$-module schemes is given in [HS19, § 4].

Conversely, let $G = \operatorname{Spec} A$ be a finite locally free strict $\mathbb{F}_\varepsilon$-module scheme over R. Note that on the additive group scheme $\mathbb{G}_{a,R} = \operatorname{Spec} R[x]$ the elements $b \in R$ act via endomorphisms $\psi_b\colon \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ given by $\psi_b^*\colon R[x] \to R[x]$, $x \mapsto bx$. This makes $\mathbb{G}_{a,R}$ into an R-module scheme, and in particular, into an $\mathbb{F}_\varepsilon$-module scheme via $\mathbb{F}_\varepsilon \subset R$. We associate with G the R-module of $\mathbb{F}_\varepsilon$-equivariant homomorphisms on R

$$\hat{M}_{\hat{q}}(G) \;:=\; \operatorname{Hom}_{R\text{-groups},\mathbb{F}_\varepsilon\text{-lin}}(G, \mathbb{G}_{a,R}) \;=\; \big\{ x \in A\colon \Delta(x) = x \otimes 1 + 1 \otimes x,\ [a](x) = ax,\ \forall a \in \mathbb{F}_\varepsilon \big\},$$

with its action of R via $R \to \operatorname{End}_{R\text{-groups},\mathbb{F}_\varepsilon\text{-lin}}(\mathbb{G}_{a,R})$. It is a finite locally free R-module by [Pog17, Proposition 3.6 and Remark 5.5]; see also [SGA 3, $\mathrm{VII_A}$, 7.4.3] in the reedited version of SGA 3 by P. Gille and P. Polo. The composition on the left with the relative $\hat{q}$-Frobenius endomorphism $F_{\hat{q},\mathbb{G}_{a,R}}$ of $\mathbb{G}_{a,R} = \operatorname{Spec} R[x]$ given by $x \mapsto x^{\hat{q}}$ defines a map $\hat{M}_{\hat{q}}(G) \to \hat{M}_{\hat{q}}(G), m \mapsto F_{\hat{q},\mathbb{G}_{a,R}} \circ m$ which is not R-linear, but $\hat{\sigma}$-linear, because $F_{\hat{q},\mathbb{G}_{a,R}} \circ \psi_b = \psi_{b^{\hat{q}}} \circ F_{\hat{q},\mathbb{G}_{a,R}}$. Therefore, $F_{\hat{q},\mathbb{G}_{a,R}}$ induces an R-homomorphism $\tau_{\hat{M}_{\hat{q}}(G)}\colon \hat{\sigma}^*\hat{M}_{\hat{q}}(G) \to \hat{M}_{\hat{q}}(G)$. Then $\underline{\hat{M}}_{\hat{q}}(G) := \big(\hat{M}_{\hat{q}}(G), \tau_{\hat{M}_{\hat{q}}(G)}\big)$ is a finite $\mathbb{F}_\varepsilon$-shtuka over R. If $f\colon G \to H$ is a morphism of finite locally free strict $\mathbb{F}_\varepsilon$-module schemes over R, then $\underline{\hat{M}}_{\hat{q}}(f)\colon \underline{\hat{M}}_{\hat{q}}(H) \to \underline{\hat{M}}_{\hat{q}}(G)$, $m \mapsto m \circ f$. This defines the functor $\underline{\hat{M}}_{\hat{q}}$ from the category of finite locally free strict $\mathbb{F}_\varepsilon$-module schemes over R to finite $\mathbb{F}_\varepsilon$-shtukas over R. It has the following properties.

Theorem 3.5 ([HS19, Theorem 5.2])**.** *(a) The contravariant functors* $\mathrm{Dr}_{\hat{q}}$ *and* $\underline{\hat{M}}_{\hat{q}}$ *are mutually quasi-inverse anti-equivalences between the category of finite* $\mathbb{F}_\varepsilon$*-shtukas over* R *and the category of finite locally free strict* $\mathbb{F}_\varepsilon$*-module schemes over* R*.*

(b) Both functors are $\mathbb{F}_q$*-linear and map short exact sequences to short exact sequences. They preserve étale objects and map the canonical decompositions from Propositions 3.2 and 3.6 below to each other.*

Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ *be a finite* $\mathbb{F}_\varepsilon$*-shtuka over* R *and let* $G = \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$*. Then*

(c) the $\mathbb{F}_\varepsilon$*-module scheme* $\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$ *is radicial over* R *if and only if* $\tau_{\hat{M}}$ *is nilpotent.*

(d) the order of the R*-group scheme* $\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$ *is* $\hat{q}^{\operatorname{rk}\underline{\hat{M}}}$*.*

(e) There is a canonical isomorphism between $\operatorname{coker} \tau_{\hat{M}} = \hat{M}/\tau_{\hat{M}}(\hat{\sigma}^*\hat{M})$ *and the co-Lie module* $\omega_{\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})} := e^*\Omega^1_{\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})/R}$ *where* $e\colon \operatorname{Spec} R \to \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$ *is the zero section.*

Proposition 3.6 ([HS19, Proposition 4.2])**.** *If* $R = L$ *is a field every* $\mathbb{F}_\varepsilon$*-module scheme* G *over* L *is canonically an extension* $0 \to G^\circ \to G \to G^{\text{ét}} \to 0$ *of an étale* $\mathbb{F}_\varepsilon$*-module scheme* $G^{\text{ét}}$ *by a connected* $\mathbb{F}_\varepsilon$*-module scheme* G°*. The* $\mathbb{F}_\varepsilon$*-module scheme* $G^{\text{ét}}$ *is the largest étale quotient of* G*. If* L *is perfect,* $G^{\text{ét}}$ *is canonically isomorphic to the reduced closed* $\mathbb{F}_\varepsilon$*-module subscheme* G^{red} *of* G *and the extension splits canonically,* $G = G^\circ \times_L G^{\text{red}}$*.*

4 Divisible local Anderson modules

Let $R \in \mathcal{N}ilp_{A_\varepsilon}$ and let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ be an effective local shtuka over R. Set $\underline{\hat{M}}_n := (\hat{M}_n, \tau_{\hat{M}_n}) := (\hat{M}/z^n\hat{M}, \tau_{\hat{M}} \bmod z^n)$ and consider the finite locally free strict $\mathbb{F}_\varepsilon$-module scheme $\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}_n)$ over R

from the previous section. $\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}_n)$ inherits from $\underline{\hat{M}}_n$ an action of $A_\varepsilon/(z^n) = \mathbb{F}_\varepsilon[z]/(z^n)$. The canonical epimorphisms $\underline{\hat{M}}_{n+1} \twoheadrightarrow \underline{\hat{M}}_n$ induce closed immersions $i_n\colon \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}_n) \hookrightarrow \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}_{n+1})$. The inductive limit $\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}) := \varinjlim \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}_n)$ in the category of sheaves on the big *fppf*-site of $\operatorname{Spec} R$ is a sheaf of A_ε-modules that satisfies the following

Definition 4.1. A *z-divisible local Anderson module over R* is a sheaf of A_ε-modules G on the big *fppf*-site of $\operatorname{Spec} R$ such that

(a) G is *z-torsion*, that is $G = \varinjlim G[z^n]$, where $G[z^n] := \ker(z^n\colon G \to G)$,

(b) G is *z-divisible*, that is $z\colon G \to G$ is an epimorphism,

(c) For every n the $\mathbb{F}_\varepsilon$-module $G[z^n]$ is representable by a finite locally free strict $\mathbb{F}_\varepsilon$-module scheme over R in the sense of Faltings [Fal02] and Abrashkin [Abr06], and

(d) locally on R there exist an integer $d \in \mathbb{Z}_{\geq 0}$, such that $(z-\zeta)^d = 0$ on ω_G where $\omega_G := \varprojlim \omega_{G[z^n]}$ and $\omega_{G[z^n]} = e^*\Omega^1_{G[z^n]/R}$ is the pullback under the zero section $e\colon \operatorname{Spec} R \to G[z^n]$. Here the action of z on ω_G comes from the structure of A_ε-module on G, while the action of ζ on ω_G comes from the structure of R-module on ω_G.

A *morphism of z-divisible local Anderson modules over R* is a morphism of *fppf*-sheaves of $\mathbb{F}_\varepsilon[\![z]\!]$-modules. It is shown in [HS19, Lemma 8.2 and Theorem 10.8] that ω_G is a finite locally free R-module and we define the *dimension of* G as $\operatorname{rk} \omega_G$. Moreover, it follows from [HS19, Proposition 7.5] that there is a locally constant function $h\colon \operatorname{Spec} R \to \mathbb{N}_0, s \mapsto h(s)$ such that the order of $G[z^n]$ equals $\hat{q}^{nh}$. We call h the *height* of the z-divisible local Anderson module G.

The category of z-divisible local Anderson modules over R and the category of local shtukas over R are both A_ε-linear. The construction and the equivalence from Section 3 extends to an equivalence between the category of effective local shtukas over R and the category of z-divisible local Anderson modules over R.

The quasi-inverse functor to $\underline{\hat{M}} \mapsto \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$ is given as follows. Let $G = \varinjlim G[z^n]$ be a z-divisible local Anderson module over R. We set

$$\underline{\hat{M}}_{\hat{q}}(G) \;=\; \big(\hat{M}_{\hat{q}}(G), \tau_{\hat{M}_{\hat{q}}(G)}\big) \;:=\; \varprojlim_n \big(\hat{M}_{\hat{q}}(G[z^n]), \tau_{\hat{M}_{\hat{q}}(G[z^n])}\big)\,.$$

Multiplication with z on G gives $\hat{M}_{\hat{q}}(G)$ the structure of an $R[\![z]\!]$-module. The following theorem was proved in [HS19, Theorem 8.3].

Theorem 4.2. *Let $R \in \mathcal{N}ilp_{A_\varepsilon}$.*

(a) The two contravariant functors $\mathrm{Dr}_{\hat{q}}$ and $\underline{\hat{M}}_{\hat{q}}$ are mutually quasi-inverse anti-equivalences between the category of effective local shtukas over R and the category of z-divisible local Anderson modules over R.

(b) Both functors are A_ε-linear, map short exact sequences to short exact sequences, and preserve (ind-) étale objects.

Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ be an effective local shtuka over R and let $G = \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$ be its associated z-divisible local Anderson module. Then

(c) G is a formal A_ε-module, i.e. a formal Lie group equipped with an action of A_ε, if and only if $\tau_{\hat{M}}$ is topologically nilpotent.

(d) the height and dimension of G are equal to the rank and dimension of $\underline{\hat{M}}$.

(e) the $R[\![z]\!]$-modules $\omega_{\mathrm{Dr}_{\hat q}(\underline{\hat M})}$ and $\operatorname{coker}\tau_{\hat M}$ are canonically isomorphic.

Example 4.3. In the notation of Example 2.3 let $R \in \mathcal{N}ilp_{A_\varepsilon}$ and let r be a positive integer. A *Drinfeld A-module of rank r over R* is a pair $\underline{E} = (E,\varphi)$ consisting of a smooth affine group scheme E over $\operatorname{Spec} R$ of relative dimension 1 and a ring homomorphism $\varphi\colon A \to \operatorname{End}_{R\text{-groups}}(E)$, $a \mapsto \varphi_a$ satisfying the following conditions:

(a) Zariski-locally on $\operatorname{Spec} R$ there is an isomorphism $\alpha\colon E \xrightarrow{\sim} \mathbb{G}_{a,R}$ of $\mathbb{F}_q$-module schemes such that

(b) the coefficients of $\Phi_a := \alpha \circ \varphi_a \circ \alpha^{-1} = \sum_{i\ge 0} b_i(a)\tau^i \in \operatorname{End}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\mathbb{G}_{a,R}) = R\{\tau\}$ satisfy $b_0(a) = \gamma(a)$, $b_{r(a)}(a) \in R^\times$ and $b_i(a)$ is nilpotent for all $i > r(a) := -r\,[\mathbb{F}_\infty : \mathbb{F}_q]\operatorname{ord}_\infty(a)$.

Here $R\{\tau\} := \big\{ \sum_{i=0}^{n} b_i\tau^i \colon n \in \mathbb{N}_0, b_i \in R \big\}$ is the non-commutative polynomial ring with $\tau b = b^q\tau$, and the isomorphism of rings $R\{\tau\} \xrightarrow{\sim} \operatorname{End}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\mathbb{G}_{a,R})$ is given by sending τ to the relative $\hat q$-Frobenius endomorphism $F_{\hat q,\mathbb{G}_{a,R}}$ of $\mathbb{G}_{a,R} = \operatorname{Spec} R[x]$ given by $x \mapsto x^{\hat q}$ and $b \in R$ to the endomorphism ψ_b given by $\psi_b^*\colon x \mapsto bx$.

For a Drinfeld A-module $\underline{E} = (E,\varphi)$ we consider the set $M := M(\underline{E}) := \operatorname{Hom}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(E,\mathbb{G}_{a,R})$ of $\mathbb{F}_q$-equivariant homomorphisms of R-group schemes. It is a locally free module over $A_R := A\otimes_{\mathbb{F}_q} R$ of rank r under the action given on $m \in M$ by

$$A \ni a\colon \quad M \longrightarrow M, \quad m \mapsto m\circ\varphi_a \quad =:\ am$$
$$R \ni b\colon \quad M \longrightarrow M, \quad m \mapsto \psi_b\circ m \quad =:\ bm$$

In addition we consider the map $\tau\colon m \mapsto F_{q,\mathbb{G}_{a,R}}\circ m$ on $m \in M$, where $F_{q,\mathbb{G}_{a,R}}$ is the relative q-Frobenius of $\mathbb{G}_{a,R}$ over R. Since $F_{q,\mathbb{G}_{a,R}}\circ\psi_b = \psi_{b^q}\circ F_{q,\mathbb{G}_{a,R}}$, and hence $\tau(bm) = b^q\tau(m)$, the map τ is σ-semilinear and induces an A_R-linear map $\tau_M\colon \sigma^*M \to M$, which makes $\underline{M}(\underline{E}) := \big(M(\underline{E}),\tau_M\big)$ into an effective A-motive over R in the sense of Example 2.3. The functor $\underline{E} \mapsto \underline{M}(\underline{E})$ is fully faithful and its essential image is described in [Har19, Theorems 3.5 and 3.9] generalizing Anderson's description [And86, Theorem 1].

Now let $\underline{\hat M} := \underline{\hat M}_\varepsilon(\underline{M}(\underline{E}))$ be the effective local $\hat\sigma$-shtuka at ε associated with $\underline{M}(\underline{E})$; see Example 2.3. Let $n \in \mathbb{N}$ and let $\varepsilon^n = (a_1,\dots,a_s) \subset A$. Then

$$\underline{E}[\varepsilon^n] \ :=\ \ker\big(\varphi_{a_1,\dots,a_s} := (\varphi_{a_1},\dots,\varphi_{a_s})\colon E \longrightarrow E^s\big)$$

is called the *ε^n-torsion submodule of $\underline{E}$*. It is an A/ε^n-module via $A/\varepsilon^n \to \operatorname{End}_R(\underline{E}[\varepsilon^n])$, $\bar a \mapsto \varphi_a$ and independent of the set of generators of ε^n; see [Har19, Lemma 6.2]. Moreover, by [Har19, Theorem 7.6] it is a finite locally free R-group scheme and a strict $\mathbb{F}_\varepsilon$-module scheme and there are canonical A/ε^n-equivariant isomorphisms of finite locally free R-group schemes

$$\mathrm{Dr}_{\hat q}(\underline{\hat M}/\varepsilon^n\underline{\hat M}) \ \xrightarrow{\ \sim\ }\ \underline{E}[\varepsilon^n] \qquad \text{and}$$
$$\underline{\hat M}/\varepsilon^n\underline{\hat M} \ \xrightarrow{\ \sim\ }\ \operatorname{Hom}_{R\text{-groups},\mathbb{F}_\varepsilon\text{-lin}}\big(\underline{E}[\varepsilon^n]\,,\,\mathbb{G}_{a,R}\big)$$

of finite $\mathbb{F}_\varepsilon$-shtukas. In particular, $\underline{E}[\varepsilon^\infty] := \varinjlim \underline{E}[\varepsilon^n] = \mathrm{Dr}_{\hat q}(\underline{\hat M})$ is a z-divisible local Anderson module over R.

5 Cohomology Realizations of Local Shtukas

In this section we work over a valuation ring $\mathcal{O}_K$ as in (b). With local shtukas over $\mathcal{O}_K$ one can associate various cohomology realizations, which are related to each other under period isomorphisms. We describe the ε-adic, the de Rham, and the crystalline realizations. These period isomorphisms are used in [HS20, HS18] to study the periods of A-motives with complex multiplication.

Definition 5.1. Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ be a local shtuka over a valuation ring $\mathcal{O}_K$ as in (b). Then $\tau_{\hat{M}}$ induces an isomorphism $\tau_{\hat{M}} \colon \hat{\sigma}^*\hat{M} \otimes_{\mathcal{O}_K[\![z]\!]} K[\![z]\!] \xrightarrow{\sim} \hat{M} \otimes_{\mathcal{O}_K[\![z]\!]} K[\![z]\!]$, because $z - \zeta \in K[\![z]\!]^\times$. We define the (*dual*) *Tate module*

$$\mathrm{H}^1_\varepsilon(\underline{\hat{M}}, A_\varepsilon) \;:=\; \check{T}_\varepsilon \underline{\hat{M}} \;:=\; (\hat{M} \otimes_{\mathcal{O}_K[\![z]\!]} K^{\mathrm{sep}}[\![z]\!])^{\hat{\tau}} \;:=\; \big\{ m \in \hat{M} \otimes_{\mathcal{O}_K[\![z]\!]} K^{\mathrm{sep}}[\![z]\!] \colon \tau_{\hat{M}}(\hat{\sigma}^*_{\hat{M}} m) = m \big\}$$

and the *rational* (*dual*) *Tate module*

$$\mathrm{H}^1_\varepsilon(\underline{\hat{M}}, Q_\varepsilon) \;:=\; \check{V}_\varepsilon \underline{\hat{M}} \;:=\; \big\{ m \in \hat{M} \otimes_{\mathcal{O}_K[\![z]\!]} K^{\mathrm{sep}}(\!(z)\!) \colon \tau_{\hat{M}}(\hat{\sigma}^*_{\hat{M}} m) = m \big\} \;=\; \check{T}_\varepsilon \underline{\hat{M}} \otimes_{A_\varepsilon} Q_\varepsilon\,.$$

By [HK20, Proposition 4.2] the Tate modules are free over A_ε, resp. Q_ε of rank equal to $\operatorname{rk} \underline{\hat{M}}$ and carry a continuous action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. They are also called the *ε-adic realizations of $\underline{\hat{M}}$.*

Theorem 5.2 ([HK20, Theorem 4.20])**.** *Assume that $\mathcal{O}_K$ is discretely valued. Then the functor $\check{T}_\varepsilon \colon \underline{\hat{M}} \mapsto \check{T}_\varepsilon \underline{\hat{M}}$ from the category of local shtukas over $\mathcal{O}_K$ to the category* $\mathrm{Rep}_{A_\varepsilon} \mathrm{Gal}(K^{\mathrm{sep}}/K)$ *of continuous representations of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ *on finite free A_ε-modules and the functor $\check{V}_\varepsilon \colon \underline{\hat{M}} \mapsto \check{V}_\varepsilon \underline{\hat{M}}$ from the category of local shtukas over $\mathcal{O}_K$ with quasi-morphisms to the category* $\mathrm{Rep}_{Q_\varepsilon} \mathrm{Gal}(K^{\mathrm{sep}}/K)$ *of continuous representations of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ *on finite dimensional Q_ε-vector spaces are fully faithful.*

Definition 5.3. Let $\mathcal{O}_K$ be discretely valued. The full subcategory of $\mathrm{Rep}_{Q_\varepsilon} \mathrm{Gal}(K^{\mathrm{sep}}/K)$ which is the essential image of the functor $\check{V}_\varepsilon$ from Theorem 5.2 is called the *category of equal characteristic crystalline representations.*

We will explain the motivation for this definition in Section 6.

Example 5.4. We describe the ε-adic (dual) Tate module $\check{T}_\varepsilon \underline{M} = \check{T}_\varepsilon \underline{\hat{M}}_\varepsilon(\underline{M})$ of the Carlitz motive $\underline{M} = (\mathcal{O}_K[t], t - \theta)$ from Example 2.5 by using the local shtuka $\underline{\hat{M}} := \underline{\hat{M}}_\varepsilon(\underline{M}) = (\mathcal{O}_K[\![z]\!], z - \zeta)$ computed there. For all $i \in \mathbb{N}_0$ let $\ell_i \in K^{\mathrm{sep}}$ be solutions of the equations $\ell_0^{\hat{q}-1} = -\zeta$ and $\ell_i^{\hat{q}} + \zeta \ell_i = \ell_{i-1}$. This implies $|\ell_i| = |\zeta|^{\hat{q}^{-i}/(\hat{q}-1)} < 1$. Define the power series $\ell_+ = \sum_{i=0}^\infty \ell_i z^i \in \mathcal{O}_{K^{\mathrm{sep}}}[\![z]\!]$. It satisfies $\hat{\sigma}(\ell_+) = (z - \zeta) \cdot \ell_+$, but depends on the choice of the ℓ_i. A different choice yields a different power series $\tilde{\ell}^+$ which satisfies $\tilde{\ell}^+ = u \ell_+$ for a unit $u \in (K^{\mathrm{sep}}[\![z]\!]^\times)^{\hat{\sigma} = \mathrm{id}} = A_\varepsilon^\times$, because $\hat{\sigma}(u) = \frac{\hat{\sigma}(\tilde{\ell}^+)}{\hat{\sigma}(\ell_+)} = \frac{\tilde{\ell}^+}{\ell_+} = u$. The field extension $\mathbb{F}_\varepsilon(\!(\zeta)\!)(\ell_i \colon i \in \mathbb{N}_0)$ of $\mathbb{F}_\varepsilon(\!(\zeta)\!)$ is the function field analog of the cyclotomic tower $\mathbb{Q}_p(\sqrt[p^i]{1} \colon i \in \mathbb{N}_0)$; see [Har09, § 1.3 and § 3.4]. There is an isomorphism of topological groups called the *ε-adic cyclotomic character*

$$\chi_\varepsilon \colon \mathrm{Gal}\big(\mathbb{F}_\varepsilon(\!(\zeta)\!)(\ell_i \colon i \in \mathbb{N}_0)/\mathbb{F}_\varepsilon(\!(\zeta)\!)\big) \;\xrightarrow{\sim}\; A_\varepsilon^\times\,,$$

which satisfies $g(\ell_+) := \sum_{i=0}^\infty g(\ell_i) z^i = \chi_\varepsilon(g) \cdot \ell_+$ in $K^{\mathrm{sep}}[\![z]\!]$ for g in the Galois group. It is independent of the choice of the ℓ_i. The ε-adic (dual) Tate module $\check{T}_\varepsilon \underline{\hat{M}}$ of $\underline{\hat{M}}$ and $\underline{M}$ is generated by ℓ_+^{-1} on which the Galois group acts by the inverse of the cyclotomic character.

Definition 5.5. Let $\underline{\hat{M}}$ be a local shtuka over a valuation ring $\mathcal{O}_K$ as in (b). We denote by $K[\![z - \zeta]\!]$ the power series ring over K in the "variable" $z - \zeta$ and by $K(\!(z - \zeta)\!)$ its fraction field. We consider the ring homomorphism $\mathcal{O}_K[\![z]\!] \hookrightarrow K[\![z - \zeta]\!]$, $z \mapsto z = \zeta + (z - \zeta)$ and define the *de Rham realization of $\underline{\hat{M}}$* as

$$\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K[\![z - \zeta]\!]\big) \;&:=\; \hat{\sigma}^*\hat{M} \otimes_{\mathcal{O}_K[\![z]\!]} K[\![z - \zeta]\!]\,, \\
\mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K(\!(z - \zeta)\!)\big) \;&:=\; \hat{\sigma}^*\hat{M} \otimes_{\mathcal{O}_K[\![z]\!]} K(\!(z - \zeta)\!) \qquad \text{and} \\
\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K) \;&:=\; \hat{\sigma}^*\hat{M} \otimes_{\mathcal{O}_K[\![z]\!],\, z \mapsto \zeta} K \\
&=\; \mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K[\![z - \zeta]\!]\big) \otimes_{K[\![z - \zeta]\!]} K[\![z - \zeta]\!]/(z - \zeta)\,.
\end{aligned}$$

The de Rham realization $\mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K(\!(z-\zeta)\!)\big)$ contains a full $K[\![z-\zeta]\!]$-lattice

$$\mathfrak{q}^{\underline{\hat{M}}} \;:=\; \tau_{\hat{M}}^{-1}(\hat{M}\otimes_{\mathcal{O}_K[\![z]\!]} K[\![z-\zeta]\!]), \tag{5.1}$$

which is called the *Hodge-Pink lattice of* $\underline{\hat{M}}$. The de Rham realization $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K)$ carries a descending separated and exhausting filtration $F^\bullet$ by K-subspaces called the *Hodge-Pink filtration of* $\underline{\hat{M}}$. It is defined via $\mathfrak{p} := \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K[\![z-\zeta]\!])$ and (for $i\in\mathbb{Z}$)

$$F^i\,\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K) \;:=\; \big(\mathfrak{p}\cap(z-\zeta)^i\mathfrak{q}^{\underline{\hat{M}}}\big)/\big((z-\zeta)\mathfrak{p}\cap(z-\zeta)^i\mathfrak{q}^{\underline{\hat{M}}}\big) \;\subset\; \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K)\,. \tag{5.2}$$

If we equip $\mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K(\!(z-\zeta)\!)\big)$ with the descending filtration $F^i\,\mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K(\!(z-\zeta)\!)\big) := (z-\zeta)^i\mathfrak{q}^{\underline{\hat{M}}}$ by $K[\![z-\zeta]\!]$-submodules, then $F^i\,\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K)$ is the image of $\mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K[\![z-\zeta]\!]\big)\cap F^i\,\mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K(\!(z-\zeta)\!)\big)$ in $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K)$. Since $z=\zeta+(z-\zeta)$ is invertible in $K[\![z-\zeta]\!]$ the de Rham realization with Hodge-Pink lattice and filtration is a functor on the category of local shtukas over $\mathcal{O}_K$ with quasi-morphisms.

Note however, that the Hodge-Pink filtration on $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K)$ does not behave well under tensor products, as opposed to the Hodge-Pink lattice; see Remark 6.3 below. Therefore, the more important concept is the Hodge-Pink lattice $\mathfrak{q}^{\underline{\hat{M}}}$.

Theorem 5.6 ([HK20, Theorem 4.15])**.** *Let $\overline{K}$ be the completion of an algebraic closure K^{alg} of K. There is a canonical functorial comparison isomorphism*

$$h_{\varepsilon,\mathrm{dR}}\colon\ \mathrm{H}^1_\varepsilon(\underline{\hat{M}}, Q_\varepsilon)\otimes_{Q_\varepsilon}\overline{K}(\!(z-\zeta)\!) \;\xrightarrow{\ \sim\ }\; \mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, K(\!(z-\zeta)\!)\big)\otimes_{K(\!(z-\zeta)\!)}\overline{K}(\!(z-\zeta)\!)\,,$$

which satisfies $h_{\varepsilon,\mathrm{dR}}\big(\mathrm{H}^1_\varepsilon(\underline{\hat{M}}, Q_\varepsilon)\otimes_{Q_\varepsilon}\overline{K}[\![z-\zeta]\!]\big) = \mathfrak{q}^{\underline{\hat{M}}}\otimes_{K[\![z-\zeta]\!]}\overline{K}[\![z-\zeta]\!]$ and which is equivariant for the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, where on the source of $h_{\varepsilon,\mathrm{dR}}$ this group acts on both factors of the tensor product and on the target of $h_{\varepsilon,\mathrm{dR}}$ it acts only on $\overline{K}$.

Definition 5.7. Let $k=\mathcal{O}_K/\mathfrak{m}_K$ be the residue field of $\mathcal{O}_K$. A *z-isocrystal* over k is a pair (D,τ_D) consisting of a finite dimensional $k(\!(z)\!)$-vector space together with a $k(\!(z)\!)$-isomorphism $\tau_D\colon \hat{\sigma}^*D \xrightarrow{\sim} D$. A *morphism* $(D,\tau_D)\to(D',\tau_{D'})$ is a $k(\!(z)\!)$-homomorphism $f\colon D\to D'$ satisfying $\tau_{D'}\circ\hat{\sigma}^*f = f\circ\tau_D$.

Definition 5.8. Let $\underline{\hat{M}} = (\hat{M},\tau_{\hat{M}})$ be local shtuka over a valuation ring $\mathcal{O}_K$ as in (b). Then the *crystalline realization* of $\underline{\hat{M}}$ is defined as the z-isocrystal over $k=\mathcal{O}_K/\mathfrak{m}_K$

$$\mathrm{H}^1_{\mathrm{cris}}\big(\underline{\hat{M}}, k(\!(z)\!)\big) \;:=\; \hat{\sigma}^*(\hat{M},\tau_{\hat{M}})\otimes_{\mathcal{O}_K[\![z]\!]} k(\!(z)\!)\,. \tag{5.3}$$

It only depends on the special fiber $\underline{\hat{M}}\otimes_{\mathcal{O}_K} k$ of $\underline{\hat{M}}$ and defines a functor $\underline{\hat{M}}\mapsto \mathrm{H}^1_{\mathrm{cris}}\big(\underline{\hat{M}}, k(\!(z)\!)\big)$ from the category of local shtukas over $\mathcal{O}_K$ with quasi-morphism to the category of z-isocrystals. This functor is faithful by [HK20, Lemma 4.24] if $\bigcap_n \hat{\sigma}^n(\mathfrak{m}_K)=(0)$.

To formulate the comparison between the de Rham and the crystalline realization we assume that there exists a fixed section $k\hookrightarrow\mathcal{O}_K$. Then there is a ring homomorphism

$$k(\!(z)\!)\longrightarrow K[\![z-\zeta]\!]\,,\quad z\longmapsto \zeta+(z-\zeta)\,,\quad \sum_i b_iz^i\longmapsto \sum_{j=0}^{\infty}(z-\zeta)^j\cdot\sum_i \tbinom{i}{j}b_i\zeta^{i-j}\,. \tag{5.4}$$

We always consider $K[\![z-\zeta]\!]$ and its fraction field $K(\!(z-\zeta)\!)$ as $k(\!(z)\!)$-vector spaces via (5.4).

Theorem 5.9 ([HK20, Theorem 5.18])**.** *Let $\underline{\hat{M}}$ be a local shtuka over $\mathcal{O}_K$. Assume that $\mathcal{O}_K$ is discretely valued or that $\underline{\hat{M}} = \underline{\hat{M}}_\varepsilon(\underline{M})$ for an A-motive $\underline{M}$ over $\mathcal{O}_K$ as in Example 2.3. Then there are canonical functorial comparison isomorphisms between the de Rham and crystalline realizations*

$$\begin{aligned} h_{\mathrm{dR,cris}}\colon\quad \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K[\![z-\zeta]\!]) &\xrightarrow{\ \sim\ } \mathrm{H}^1_{\mathrm{cris}}\big(\underline{\hat{M}}, k(\!(z)\!)\big)\otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]\quad \textit{and}\\ h_{\mathrm{dR,cris}}\colon\quad \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, K) &\xrightarrow{\ \sim\ } \mathrm{H}^1_{\mathrm{cris}}\big(\underline{\hat{M}}, k(\!(z)\!)\big)\otimes_{k(\!(z)\!),\,z\mapsto\zeta} K\,.\end{aligned}$$

To formulate the comparison between the crystalline and the ε-adic realizations we introduce the $\mathcal{O}_K$-algebra

$$\mathcal{O}_K[\![z,z^{-1}\}\ \ :=\ \ \Big\{\sum_{i=-\infty}^{\infty} b_i z^i\colon\ b_i\in\mathcal{O}_K\,,\ |b_i|\,|\zeta|^{ri}\to 0\ (i\to-\infty)\text{ for all }r>0\Big\}\,. \tag{5.5}$$

It is a subring of $K[\![z-\zeta]\!]$ via the expansion $\sum\limits_{i=-\infty}^{\infty} b_i z^i = \sum\limits_{j=0}^{\infty}\zeta^{-j}\Big(\sum\limits_{i=-\infty}^{\infty}\binom{i}{j}b_i\zeta^i\Big)(z-\zeta)^j$. The homomorphism (5.4) factors through $\mathcal{O}_K[\![z,z^{-1}\}$. We view the elements of $\mathcal{O}_K[\![z,z^{-1}\}$ as functions that converge on the punctured open unit disc $\{0<|z|<1\}$. An example of such a function is

$$\ell_- := \prod_{i\in\mathbb{N}_0}(1-\tfrac{\zeta^{\hat q^i}}{z})\ \in\ \mathbb{F}_\varepsilon[\![\zeta]\!][\![z,z^{-1}\}\ \subset\ \mathcal{O}_K[\![z,z^{-1}\}\,, \tag{5.6}$$

which satisfies $\ell_-=(1-\frac{\zeta}{z})\cdot\hat\sigma(\ell_-)$. In addition, we let $\overline{K}$ be the completion of an algebraic closure $K^{\rm alg}$ of K and recall the element $\ell_+\in\mathcal{O}_{\overline{K}}[\![z]\!]$ from Example 5.4, which satisfies $\hat\sigma(\ell_+)=(z-\zeta)\cdot\ell_+$. We set

$$\ell\ :=\ \ell_+\ell_-\ \in\ \mathcal{O}_{\overline{K}}[\![z,z^{-1}\}\,. \tag{5.7}$$

Then $\hat\sigma(\ell)=z\cdot\ell$ and $g(\ell)=\chi_\varepsilon(g)\cdot\ell$ for $g\in\mathrm{Gal}(K^{\rm sep}/K)$ where χ_ε is the cyclotomic character from Example 5.4.

Theorem 5.10 ([HK20, Theorem 5.20])**.** *Let $\underline{\hat M}$ be a local shtuka over $\mathcal{O}_K$. Assume that $\mathcal{O}_K$ is discretely valued or that $\underline{\hat M}=\underline{\hat M}_\varepsilon(\underline{M})$ for an A-motive $\underline{M}$ over $\mathcal{O}_K$ as in Example 2.3. Then there is a canonical functorial comparison isomorphism between the ε-adic and crystalline realizations*

$$h_{\varepsilon,\rm cris}\colon\ \ \mathrm{H}^1_\varepsilon(\underline{\hat M},Q_\varepsilon)\otimes_{Q_\varepsilon}\mathcal{O}_{\overline{K}}[\![z,z^{-1}\}[\ell^{-1}]\ \xrightarrow{\ \sim\ }\ \mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)\otimes_{k(\!(z)\!)}\mathcal{O}_{\overline{K}}[\![z,z^{-1}\}[\ell^{-1}]\,.$$

The isomorphism $h_{\varepsilon,\rm cris}$ is $\mathrm{Gal}(K^{\rm sep}/K)$- and $\hat\tau$-equivariant, where on the left module $\mathrm{Gal}(K^{\rm sep}/K)$ acts on both factors and $\hat\tau$ is $\mathrm{id}\otimes\hat\sigma$, and on the right module $\mathrm{Gal}(K^{\rm sep}/K)$ acts only on $\mathcal{O}_{\overline{K}}[\![z,z^{-1}\}[\ell^{-1}]$ and $\hat\tau$ is $(\tau_D\circ\hat\sigma_D^*)\otimes\hat\sigma$. In other words $h_{\varepsilon,\rm cris}=\tau_D\circ\hat\sigma^*h_{\varepsilon,\rm cris}$. Moreover, $h_{\varepsilon,\rm cris}$ satisfies $h_{\varepsilon,\rm dR}=(h^{-1}_{\rm dR,cris}\otimes\mathrm{id}_{\overline{K}(\!(z-\zeta)\!)})\circ(h_{\varepsilon,\rm cris}\otimes\mathrm{id}_{\overline{K}(\!(z-\zeta)\!)})$. It allows to recover $\mathrm{H}^1_\varepsilon(\underline{\hat M},Q_\varepsilon)$ from $\mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)$ as the intersection inside $\mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)\otimes_{k(\!(z)\!)}\overline{K}(\!(z-\zeta)\!)$

$$h_{\varepsilon,\rm cris}\big(\mathrm{H}^1_\varepsilon(\underline{\hat M},Q_\varepsilon)\big)\ =\ \big(\mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)\otimes_{k(\!(z)\!)}\mathcal{O}_{\overline{K}}[\![z,z^{-1}\}[\ell^{-1}]\big)^{\hat\tau=\rm id}\cap\mathfrak{q}_D\otimes_{K[\![z-\zeta]\!]}\overline{K}[\![z-\zeta]\!]\,,$$

where $\mathfrak{q}_D\subset\mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)\otimes_{k(\!(z)\!)}K(\!(z-\zeta)\!)$ is the Hodge-Pink lattice of $\underline{\hat M}$.

6 Crystalline Representations over Function Fields

We explain the motivation for Definition 5.3, compare [HK20, Remarks 5.13 and 6.17].

Let $\mathcal{O}_K$ is discretely valued and let $\underline{\hat M}$ be a local shtuka over $\mathcal{O}_K$. Theorem 5.9 allows to define a Hodge-Pink lattice and a Hodge-Pink filtration on $\mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)$. More precisely, we equip the finite dimensional $k(\!(z)\!)$-vector space $D:=\mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)$ with the Hodge-Pink lattice

$$\mathfrak{q}_D\ :=\ (h_{\rm dR,cris}\otimes\mathrm{id}_{K(\!(z-\zeta)\!)})(\mathfrak{q}^{\underline{\hat M}})\ \subset\ D\otimes_{k(\!(z)\!)}K(\!(z-\zeta)\!)\,,$$

where $\mathfrak{q}^{\underline{\hat M}}\subset\mathrm{H}^1_{\rm dR}\big(\underline{\hat M},K(\!(z-\zeta)\!)\big)$ is the Hodge-Pink lattice from (5.1). Together with the Frobenius $\tau_D:=\hat\sigma^*\tau_{\hat M}\otimes\mathrm{id}_{k(\!(z)\!)}$ on $D=\mathrm{H}^1_{\rm cris}\big(\underline{\hat M},k(\!(z)\!)\big)$ from (5.3), the triple $\underline{D}(\underline{\hat M}):=\underline{D}=(D,\tau_D,\mathfrak{q}_D)$ forms a z-isocrystal with Hodge-Pink structure as in the following

Definition 6.1. A *z-isocrystal with Hodge-Pink structure over $\mathcal{O}_K$* is a triple $\underline{D} = (D, \tau_D, \mathfrak{q}_D)$ consisting of a z-isocrystal (D, τ_D) over k and a $K[\![z-\zeta]\!]$-lattice $\mathfrak{q}_D$ in $D \otimes_{k(\!(z)\!)} K(\!(z-\zeta)\!)$ of full rank, which is called the *Hodge-Pink lattice of $\underline{D}$*. The dimension of D is called the *rank of $\underline{D}$* and is denoted $\operatorname{rk} \underline{D}$.

A *morphism* $(D, \tau_D, \mathfrak{q}_D) \to (D', \tau_{D'}, \mathfrak{q}_{D'})$ is a $k(\!(z)\!)$-homomorphism $f \colon D \to D'$ satisfying $\tau_{D'} \circ \hat{\sigma}^* f = f \circ \tau_D$ and $(f \otimes \mathrm{id})(\mathfrak{q}_D) \subset \mathfrak{q}_{D'}$.

A *strict subobject* $\underline{D}' \subset \underline{D}$ is a z-isocrystal with Hodge-Pink structure of the form $\underline{D}' = \big(D',\, \tau_D|_{\hat{\sigma}^* D'},\, \mathfrak{q}_D \cap D' \otimes_{k(\!(z)\!)} K(\!(z-\zeta)\!)\big)$ where $D' \subset D$ is a $k(\!(z)\!)$-subspace with $\tau_D(\hat{\sigma}^* D') = D'$.

On a z-isocrystal with Hodge-Pink structure $\underline{D}$ there always is the tautological $K[\![z-\zeta]\!]$-lattice $\mathfrak{p}_D := D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]$. Since $K[\![z-\zeta]\!]$ is a principal ideal domain the elementary divisor theorem provides basis vectors $v_i \in \mathfrak{p}_D$ such that $\mathfrak{p}_D = \bigoplus_{i=1}^r K[\![z-\zeta]\!] \cdot v_i$ and $\mathfrak{q}_D = \bigoplus_{i=1}^r K[\![z-\zeta]\!] \cdot (z-\zeta)^{\mu_i} \cdot v_i$ for integers $\mu_1 \geq \ldots \geq \mu_r$. We call $(\mu_1, \ldots, \mu_r)$ the *Hodge-Pink weights of $\underline{D}$*. Alternatively if e is large enough such that $\mathfrak{q}_D \subset (z-\zeta)^{-e}\mathfrak{p}_D$ or $(z-\zeta)^e \mathfrak{p}_D \subset \mathfrak{q}_D$ then the Hodge-Pink weights are characterized by

$$\begin{aligned} (z-\zeta)^{-e}\mathfrak{p}_D/\mathfrak{q}_D &\cong \bigoplus_{i=1}^n K[\![z-\zeta]\!]/(z-\zeta)^{e+\mu_i}\,, \\ \text{or} \quad \mathfrak{q}_D/(z-\zeta)^e\mathfrak{p}_D &\cong \bigoplus_{i=1}^n K[\![z-\zeta]\!]/(z-\zeta)^{e-\mu_i} \end{aligned}$$

Like in (5.2) the Hodge-Pink lattice $\mathfrak{q}_D$ induces a descending filtration of $D_K := D \otimes_{k(\!(z)\!),\, z \mapsto \zeta} K$ by K-subspaces as follows. Consider the natural projection

$$\mathfrak{p}_D \twoheadrightarrow \mathfrak{p}_D/(z-\zeta)\mathfrak{p}_D = D_K\,.$$

The *Hodge-Pink filtration* $F^\bullet D_K = (F^i D_K)_{i \in \mathbb{Z}}$ is defined by letting $F^i D_K$ be the image in D_K of $\mathfrak{p}_D \cap (z-\zeta)^i \mathfrak{q}_D$ for all $i \in \mathbb{Z}$. This means, $F^i D_K = \big(\mathfrak{p}_D \cap (z-\zeta)^i\mathfrak{q}_D\big) / \big((z-\zeta)\mathfrak{p}_D \cap (z-\zeta)^i \mathfrak{q}_D\big)$.

Definition 6.2. Let $\underline{D} = (D, \tau_D, \mathfrak{q}_D)$ be a z-isocrystal with Hodge-Pink structure over $\mathcal{O}_K$ and set $r = \dim_{k(\!(z)\!)} D$.

(a) Choose a $k(\!(z)\!)$-basis of D and let $\det \tau_D$ be the determinant of the matrix representing τ_D with respect to this basis. The number $t_N(\underline{D}) := \operatorname{ord}_z(\det \tau_D)$ is independent of this basis and is called the *Newton slope of $\underline{D}$*.

(b) The integer $t_H(\underline{D}) := -\mu_1 - \ldots - \mu_r$, where $\mu_1, \ldots, \mu_r$ are the Hodge-Pink weights of $\underline{D}$ from Definition 6.1, satisfies $\wedge^r \mathfrak{q}_D = (z-\zeta)^{-t_H(\underline{D})} \wedge^r \mathfrak{p}_D$ and is called the *Hodge slope of $\underline{D}$*.

(c) $\underline{D}$ is called *weakly admissible* if

$$t_H(\underline{D}) = t_N(\underline{D}) \qquad \text{and} \qquad t_H(\underline{D}') \leq t_N(\underline{D}') \quad \text{for every strict subobject } \underline{D}' \subset \underline{D}.$$

Remark 6.3. One can show that the tensor product

$$\underline{D} \otimes \underline{D}' = \big(D \otimes_{k(\!(z)\!)} D', \tau_D \otimes \tau_{D'}, \mathfrak{q}_D \otimes_{K[\![z-\zeta]\!]} \mathfrak{q}_{D'}\big)$$

of two weakly admissible z-isocrystals with Hodge-Pink structures $\underline{D}$ and $\underline{D}'$ over $\mathcal{O}_K$ is again weakly admissible. It was Pink's insight that for this result the *Hodge-Pink filtration* does not suffice, but one needs the finer information present in the *Hodge-Pink lattice*. The problem arises if the field extension $K/\mathbb{F}_q(\!(\zeta)\!)$ is inseparable; see [Pin97b, Example 5.16]. This is Pink's ingenious discovery.

Proposition 6.4 ([HK20, Corollary 6.11])**.** *Let $\underline{\hat{M}}$ be a local shtuka over $\mathcal{O}_K$. Assume that $\mathcal{O}_K$ is discretely valued or that $\underline{\hat{M}} = \underline{\hat{M}}_\varepsilon(\underline{M})$ for an A-motive $\underline{M}$ over $\mathcal{O}_K$ as in Example 2.3. Then the z-isocrystal with Hodge-Pink structure $\underline{D}(\underline{\hat{M}})$ constructed at the beginning of this section is weakly admissible. The functor $\underline{\hat{M}} \longmapsto \underline{D}(\underline{\hat{M}})$ from the category of local shtukas over $\mathcal{O}_K$ with quasi-morphisms to the category of weakly admissible z-isocrystals with Hodge-Pink structure is fully faithful.*

There is a converse to this proposition.

Theorem 6.5 ([GL11, Théorème 7.3], [Har11, Theorem 2.5.3])**.** *If $\mathcal{O}_K$ is discretely valued then every weakly admissible z-isocrystal with Hodge-Pink structure $\underline{D}$ over $\mathcal{O}_K$ is of the form $\underline{D}(\underline{\hat{M}})$ for a local shtuka $\underline{\hat{M}}$ over $\mathcal{O}_K$.*

Remark 6.6. The theory presented here has as analog the theory of p-adic Galois representations. There L is a discretely valued extension of $\mathbb{Q}_p$ with perfect residue field κ and $L_0 := W(\kappa)[\frac{1}{p}]$ is the maximal, absolutely unramified subfield of L. Let $\hat{\sigma} := W(\mathrm{Frob}_p)$ be the lift to L_0 of the p-Frobenius on κ which fixes the uniformizer p of L_0. Crystalline p-adic Galois representations are described by *filtered isocrystals* $\underline{D} = (D, \tau_D, F^\bullet D_L)$ over L, where D is a finite dimensional L_0-vector space, $\tau_D\colon \hat{\sigma}^* D \xrightarrow{\sim} D$ is an L_0-isomorphism and $F^\bullet D_L$ is a descending filtration on $D_L := D \otimes_{L_0} L$ by L-subspaces. More precisely, the Theorem of Colmez and Fontaine [CF00, Théorème A] says that a continuous representation of $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ in a finite dimensional $\mathbb{Q}_p$-vector space is crystalline if and only if it isomorphic to $F^0(\underline{D} \otimes_{L_0} \widetilde{\mathbf{B}}_{\mathrm{rig}})^{\tau = \mathrm{id}}$ for a *weakly admissible* filtered isocrystal $\underline{D} = (D, \tau_D, F^\bullet D_L)$ over L. Here $\widetilde{\mathbf{B}}_{\mathrm{rig}}$ is a certain period ring from Fontaine's theory of p-adic Galois representations, which carries a filtration and a Frobenius endomorphism Frob_p. The function field analog of $\widetilde{\mathbf{B}}_{\mathrm{rig}}$ is the Q_ε-algebra $\mathcal{O}_{\overline{K}}[\![z, z^{-1}\}[\ell^{-1}]$; see [Har09, §§ 2.5 and 2.7]. In the function field case, when K is discretely valued, we could therefore define the *category of equal characteristic crystalline representations of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ as the essential image of the functor

$$\underline{D} = (D, \tau_D, \mathfrak{q}_D) \longmapsto \left(\underline{D} \otimes_{k(\!(z)\!)} \mathcal{O}_{\overline{K}}[\![z, z^{-1}\}[\ell^{-1}]\right)^{\tau = \mathrm{id}} \cap \mathfrak{q}_D \otimes_{K[\![z-\zeta]\!]} \overline{K}[\![z - \zeta]\!] \tag{6.8}$$

from weakly admissible z-isocrystals with Hodge-Pink structure $\underline{D}$ to continuous representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ in finite dimensional Q_ε-vector spaces. By Theorems 6.5, 5.10 and 5.2 and Proposition 6.4 this functor is fully faithful and this definition coincides with our Definition 5.3 above.

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